

# FINITE TRACES AND REPRESENTATIONS OF THE GROUP OF INFINITE MATRICES OVER A FINITE FIELD.

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**ABSTRACT.** The article is devoted to the representation theory of locally compact group **GLB** of almost upper-triangular infinite matrices over finite field, which we view as an adequate  $n = \infty$  analogue of general linear groups  $\mathrm{GL}(n, q)$ . We describe all semifinite traces (characters) of **GLB** which are finite on the appropriate subspace  $\mathcal{A}(\mathbf{GLB})$  of smooth functions in  $L_1$  on the group.

We further distinguish a class of unipotent traces and explore their properties, including remarkable multiplicativity and connections with conjugation-invariant probability measures on the group of upper-triangular matrices over finite field. We construct representations of **GLB** corresponding to the wide class of indecomposable unipotent traces. Our construction is based on the natural action of **GLB** in the space of flags in the countable infinite-dimensional vector space and leads to von Neumann type  $II_\infty$  factor representations.

We also study and decompose the (bi-)regular representation of **GLB**. Finally, various connections between representation theory of **GLB** and representation theory of the infinite-dimensional Iwahori-Hecke algebra  $\mathcal{H}_q(\infty)$  and infinite symmetric group  $S(\infty)$  (which is the inductive limit of symmetric groups  $S(n)$ ) are explained.

The main stream of this paper is a continuation and development of the series of the previous papers by S.Kerov and A.Vershik of 70-90s on the representation theory of finite and infinite symmetric groups. The topics we study should be considered, in general, as a part of the so-called asymptotic representation theory.

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#### HISTORICAL PREFACE

My joint work with S. Kerov on the asymptotic representation theory of the matrix groups  $\mathbb{GL}(n, q)$  over finite field as the rank  $n$  grows to infinity, was started at the beginning of 80s as a continuation of our papers devoted to analogous problems for symmetric groups of growing ranks. The problem setup was taking into account not only the analogy between these two series of groups, but also an important distinction in the definition of the inductive limit of group algebras, which should be modified for  $\mathbb{GL}(n, q)$ . “True” (i.e. parabolic) embedding of these group algebras was well-known starting from the very first papers on the representation theory of  $\mathbb{GL}(n, q)$  (see [Gr], [Zel], [F], etc). It was used by A. Zelevinsky and us (see [V82]) to define a natural *limit object* (i.e. inductive limit) which is the group  $\mathbf{GLB}$  of infinite matrices (over finite field) with finitely many non-zero elements below the main diagonal. The results of the book [Zel] in which the representation theory of  $\mathbb{GL}(n, q)$  is studied via the Hopf algebras theory, are not directly related to the asymptotic representation theory but are substantial for the problems’ setup.

However, the project was suspended and we returned to this topic only in the middle of 90s with the first article [VK98] appearing in 1998. In this paper we gave main definitions and sketched the plan of further research. In 1998–1999 we prepared some more detailed texts. Already after the sad death of my former student and coauthor an outstanding mathematician and person S. Kerov (1946–2000) an improved version of these texts was published [VK07]. Also lecture notes [V03] based on my talks at school EMS–NATO in The Euler International Mathematical Institute in 2001 were published a bit earlier (2003). Several talks devoted to this topic were presented at various conferences and seminars. All these texts contained (mostly without proofs) a number of statements forming an initial foundation of the asymptotic representation theory of group  $\mathbf{GLB}$ .

V. Gorin following the general plan contained in [VK07] interpreted and supplemented with complete proofs the statements of that article. The present paper is

a result of this work. We can say that this article, finally, concludes the first step in the study of the representation theory of **GLB**.

Further work related to this important topic will, probably, be based on the scheme presented here and might have several directions. First, a more detailed study of the principal representations (i.e. representations induced from the Borel subgroup) is required; this part of theory is especially close to the representation theory of the infinite symmetric group. Second, we need to find a more explicit reduction of other representations, including cuspidal ones and others, to the principal representations. The most interesting questions are related to the harmonic analysis and combinatorics of the group algebra of **GLB**, to the interpretation of characters as measures, to statistics of Jordan normal forms, etc, and also to the representations of this group, both irreducible and factor representations. One can hope that this theory will be of use for the general representation theory of locally compact groups and will also extend our understanding of the representation theory of groups  $\mathrm{GL}(n, q)$ .

A. Vershik

## 1. INTRODUCTION

**1.1. Overview.** Let  $\mathbb{F}_q$  be the finite field with  $q$  elements and let  $\mathrm{GL}(n, q)$  denote the group of all invertible  $n \times n$  matrices over  $\mathbb{F}_q$ . In the present article we study the group **GLB** of all almost upper-triangular matrices (i.e. containing finitely many nonzero elements below diagonal) over  $\mathbb{F}_q$ , which we view as  $n = \infty$  analogue of groups  $\mathrm{GL}(n, q)$ , as will be explained in Section 1.6. In other words, if  $V_\infty$  is the (countable) vector space of all finite vectors in  $\mathbb{F}_q^\infty$  and  $V_n$  is the subspace spanned by the first  $n$  basis vectors, then **GLB** consists of all linear transforms of  $V_\infty$  preserving all but finitely many spaces  $V_n$ .

An important feature of the infinite-dimensional group **GLB** is its *local compactness*. **GLB** possesses the Haar measure  $\mu_{\mathbf{GLB}}$  which allows to introduce the group algebra  $L_1(\mathbf{GLB}, \mu_{\mathbf{GLB}})$  with multiplication given by the convolution. In our study we intensively use an important dense subalgebra  $\mathcal{A}(\mathbf{GLB})$  of  $L_1(\mathbf{GLB}, \mu_{\mathbf{GLB}})$  consisting of all continuous functions with compact support taking only finitely many values.

As we show the group **GLB** has a rich family of traces (characters), which might be singular on the group itself, but are finite on  $\mathcal{A}(\mathbf{GLB})$ . These traces give rise to type *II* factor representations and form a basis for the representation theory and harmonic analysis on **GLB**. They are the main object of our interest.

In the present article we concentrate on the following topics, where we prove a variety of results.

- We study the algebra  $L_1(\mathbf{GLB}, \mu_{\mathbf{GLB}})$  and its dense locally semisimple subalgebra  $\mathcal{A}(\mathbf{GLB})$ . The structure of the latter as an inductive limit of finite-dimensional algebras is explained and thoroughly investigated.
- We describe *all* finite traces of  $\mathcal{A}(\mathbf{GLB})$ .
- We distinguish the family of unipotent (principal) traces, find their remarkable properties and explain their connections with infinite-dimensional Iwahori-Hecke algebra  $\mathcal{H}_q(\infty)$ , infinite symmetric group  $S(\infty)$  and unipotent representations of  $\mathrm{GL}(n, q)$ .
- We show that each unipotent trace can be identified with conjugation-invariant ergodic probability measure on the Borel subgroup  $\mathbf{B} \subset \mathbf{GLB}$  of

all upper-triangular matrices. A number of theorems and conjectures on the structure of such measures is presented.

- We give a construction for the wide family of representations of  $\mathbf{GLB}$  based on the natural action of  $\mathbf{GLB}$  in the spaces of flags in the infinite-dimensional vector space over  $\mathbb{F}_q$  and in the *principal gruppoid* defined by this action. We prove that these representations are von Neumann type  $II_\infty$  factor representations and compute their traces, which are identified with extreme (indecomposable) unipotent traces of  $\mathcal{A}(\mathbf{GLB})$  with explicit parameters in our classification of all finite traces.
- We study the (bi-)regular representation of  $\mathbf{GLB}$ , show that it possesses a natural trace which is finite on  $\mathcal{A}(\mathbf{GLB})$  and decompose this trace into extremes.

In Sections 1.2–1.5 a more detailed description of our work is given. In Section 1.6 we motivate our definitions and some of the choices, which otherwise might seem arbitrary. We also summarize the similarities with asymptotic representation theory of symmetric groups in the same section. The brief list of key notations and theorems is given in Section 1.7. Finally, we want to remark that most of the results of the present paper were announced (without proofs) in articles [VK98], [V03], [VK07]. In this text we refine those statements and give full proofs of them, also a number of entirely new results is presented.

**1.2. Schwartz-Bruhat algebra  $\mathcal{A}(\mathbf{GLB})$  and its traces.** One of the central objects of our study is algebra  $\mathcal{A}(\mathbf{GLB}) \subset L_1(\mathbf{GLB}, \mu_{\mathbf{GLB}})$  consisting of all continuous functions on  $\mathbf{GLB}$  with compact support taking only finitely many values.  $\mathcal{A}(\mathbf{GLB})$  can be viewed as an analogue of the algebra of smooth functions or Schwartz-Bruhat algebra in the theory of linear  $p$ -adic groups, see e.g. [BZ], [GGP].

In Section 2 we show that algebra  $\mathcal{A}(\mathbf{GLB})$  is an inductive limit of group algebras  $\mathcal{A}(\mathbf{GLB})_n \simeq \mathbb{C}(\mathbb{GL}(n, q))$ . However, the arising embeddings  $i_n : \mathbb{C}(\mathbb{GL}(n, q)) \hookrightarrow \mathbb{C}(\mathbb{GL}(n+1, q))$  are not induced by group embeddings, instead we should use *parabolic embeddings*, which are averagings by certain subgroups. This description implies that  $\mathcal{A}(\mathbf{GLB})$  is locally semisimple algebra, which means, in particular, that the enveloping  $C^*$ -algebra of  $\mathbf{GLB}$  is almost finite dimensional (AF-) algebra; see e.g. [Br], [VK87], [SV], [K03].

As for every locally semisimple algebra the structure of the algebra  $\mathcal{A}(\mathbf{GLB})$  is uniquely defined by its *Bratteli diagram*. Recall that Bratteli diagram is a graded graph with vertices at level  $n$  symbolizing irreducible representations of  $\mathcal{A}(\mathbf{GLB})_n$  and edges between adjacent levels symbolizing the inclusion relations of the representations. The inclusions  $i_n : \mathcal{A}(\mathbf{GLB})_n \hookrightarrow \mathcal{A}(\mathbf{GLB})_{n+1}$  are not unital and the algebra  $\mathcal{A}(\mathbf{GLB})$  has no unit element, thus, each vertex of the Bratteli diagram is to be supplemented with additional label which is the dimension of the corresponding irreducible representation. In our case these numbers are dimensions of irreducible representations of groups  $\mathbb{GL}(n, q)$  and they admit explicit formulas (e.g. a  $q$ -analogue of the classical hook formula, see e.g. [M, Chapter IV]).

We show in Section 2 that the Bratteli diagram of  $\mathcal{A}(\mathbf{GLB})$  is a union of countably many copies of the *Young graph*  $\mathbb{Y}$  with shifted grading. Recall that level  $n$  of  $\mathbb{Y}$  consist of all Young diagrams with  $n$  boxes (equivalently, partitions of  $n$ ) with edges joining the diagrams which differ by addition of a single box. Therefore, the algebra  $\mathcal{A}(\mathbf{GLB})$  is a direct sum of countably many (non-isomorphic) ideals corresponding to different copies of  $\mathbb{Y}$ .

We also show that infinite Iwahori-Hecke algebra  $\mathcal{H}_q(\infty)$  is naturally embedded into  $\mathcal{A}(\mathbf{GLB})$ , moreover, it is a subset of one of the above ideals. This somehow explains the appearance of the Young graph. Indeed,  $\mathbb{Y}$  (without any labels, since  $\mathcal{H}_q(\infty)$  contains the unit element) is the Bratteli diagram of  $\mathcal{H}_q(\infty)$  as follows from the fact that  $\mathcal{H}_q(\infty)$  and the group algebra of  $S(\infty)$  are isomorphic, see e.g. [VK89].

We further concentrate on the representation theory of  $\mathcal{A}(\mathbf{GLB})$  and  $\mathbf{GLB}$ . It is well-known that the representation theory of “big” groups, such as  $S(\infty)$ ,  $U(\infty)$ ,  $\mathbb{GL}(\infty, q)$  is *wild* and in order to get a well-behaved theory one has to restrict the class of the representations under consideration. There are two approaches here, one of them deals with von Neumann factor representations, see e.g. [Th84], and another one studies representations of  $(G, K)$ -pairs, see e.g. [O2]. In both approaches representations are in correspondence with characters (or traces) of a group (or algebra), thus, it is crucial to obtain the classification of traces.

In Section 2.6 we describe the set of finite traces of  $\mathcal{A}(\mathbf{GLB})$ . We show (see Theorem 2.26) that this set is a simplicial cone with extreme rays parameterized by triplets  $(f, \alpha, \beta)$ , where  $\alpha = \{\alpha_i\}$  and  $\beta = \{\beta_i\}$  are non-increasing sequences of non-negative reals satisfying:

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq 0, \quad \beta_1 \geq \beta_2 \geq \cdots \geq 0, \quad \sum_{i=1}^{\infty} (\alpha_i + \beta_i) \leq 1,$$

and  $f$  is an element of a certain (explicit) countable set  $\mathcal{CY}'$ . This result is to a large extent based on the classification theorem for characters of the infinite symmetric group  $S(\infty)$  first proved by Thoma in [Th64]. For each extreme trace parameterized by  $(f, \alpha, \beta)$  we give a formula for its values on  $\mathcal{A}(\mathbf{GLB})$  in terms of the values of characters of irreducible representations of  $\mathbb{GL}(n, q)$ .

**1.3. Unipotent traces.** Among the traces of  $\mathcal{A}(\mathbf{GLB})$  we distinguish a class of *unipotent* traces, which are closely related to the same-named representations of  $\mathbb{GL}(n, q)$ . In our classification of traces the unipotent extreme ones are such that  $f = f_0$ , where  $f_0$  is a certain special element of  $\mathcal{CY}'$ . They are distinguished by the fact that the values on (at least some) elements of  $\mathcal{H}_q(\infty) \subset \mathcal{A}(\mathbf{GLB})$  are non-zero. On the contrary, if  $f \neq f_0$ , then for arbitrary  $\alpha$  and  $\beta$  the corresponding trace vanishes on  $\mathcal{H}_q(\infty)$ . Moreover, the restriction of extreme unipotent trace on  $\mathcal{A}(\mathbf{GLB})_n \simeq \mathbb{C}(\mathbb{GL}(n, q))$  is a linear combination of traces of irreducible unipotent representations of  $\mathbb{GL}(n, q)$ , see [St], [J2] for more information on unipotent representation of  $\mathbb{GL}(n, q)$ .

The extreme unipotent traces are in one-to-one correspondence with extreme traces of Iwahori-Hecke algebra  $\mathcal{H}_q(\infty)$  (and, thus, with characters of the infinite symmetric group  $S(\infty)$ ). Extreme traces of  $\mathcal{H}_q(\infty)$  are parameterized (see [VK89] and also [Me, Section 7]) by two sequences

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq 0, \quad \beta_1 \geq \beta_2 \geq \cdots \geq 0, \quad \sum_i (\alpha_i + \beta_i) \leq 1$$

and the (normalized) restriction of the extreme unipotent trace of  $\mathcal{A}(\mathbf{GLB})$  on  $\mathcal{H}_q(\infty) \subset \mathcal{A}(\mathbf{GLB})$  is extreme trace of  $\mathcal{H}_q(\infty)$  with the same parameters. This provides an infinite-dimensional analogue of the well-known correspondence between irreducible representations of  $\mathcal{H}_q(n)$  and unipotent irreducible representations of  $\mathbb{GL}(n, q)$ , see e.g. [CF].

The extreme unipotent traces have a number of intriguing properties which we discuss in Section 3.

Recall the following multiplicativity property for the characters of  $S(\infty)$ , see [Th64], [VK81], [KOO]. Let  $\chi$  be a normalized character (i.e. central positive-definite function satisfying  $\chi(e) = 1$ ) of  $S(\infty)$ . Then  $\chi$  is extreme (i.e. extreme point of the convex set of all normalized characters) if and only if the following multiplicativity property is satisfied. For  $g \in S(n) \subset S(\infty)$  and  $h \in S(m) \subset S(\infty)$  let  $g \odot h$  denote the element of  $S(n+m) \subset S(\infty)$  obtained by adjoining the permutations  $g$  and  $h$ , i.e. by making  $g$  act on  $1, \dots, n$  and  $h$  act on  $n+1, \dots, n+m$ . Then for all  $n, m, g, h$  we have

$$(1.1) \quad \chi(g \odot h) = \chi(g)\chi(h).$$

We prove (see Theorem 3.3) that extreme unipotent traces of  $\mathcal{A}(\mathbf{GLB})$  satisfy an analogue of (1.1). More precisely, let  $e_g$  denote the element of  $\mathcal{A}(\mathbf{GLB})_n \subset \mathcal{A}(\mathbf{GLB})$  corresponding to  $g$  under the identification of  $\mathcal{A}(\mathbf{GLB})_n$  and group algebra of  $\mathbf{GL}(n, q)$ . Further, for  $g \in \mathbf{GL}(n, q)$ ,  $h \in \mathbf{GL}(m, q)$  let  $g \odot h$  denote the block-diagonal matrix in  $\mathbf{GL}(n+m, q)$  with blocks  $n$  and  $m$ . Then every *normalized* extreme unipotent trace  $\chi$  of  $\mathcal{A}(\mathbf{GLB})$  satisfies

$$\chi(e_{g \odot h}) = \chi(e_g)\chi(e_h).$$

for any  $g, h$  with coprime characteristic polynomials. We further compute the values of extreme unipotent traces on arbitrary elements of  $\mathcal{A}(\mathbf{GLB})$  in terms of explicit *specializations* (i.e. homomorphisms into  $\mathbb{C}$ ) of the algebra of symmetric functions  $\Lambda$ , see Theorem 3.5.

We continue the discussion of unipotent traces by showing their relation to certain probability measures. We prove (see Theorem 4.1) that each extreme unipotent trace of  $\mathcal{A}(\mathbf{GLB})$  gives rise to a probability measure on the subgroup  $\mathbf{B} \subset \mathbf{GLB}$  of upper-triangular matrices and, moreover, if two traces are not proportional, then corresponding measures are distinct. Also each such measure on  $\mathbf{B}$  can be naturally extended to a signed (i.e. not necessary positive)  $\sigma$ -finite measure on  $\mathbf{GLB}$  invariant under conjugations. This gives an interpretation of traces of  $\mathcal{A}(\mathbf{GLB})$  as characters of the group  $\mathbf{GLB}$  which are infinite (i.e. not well-defined) on the group itself, but have a singularity of type measure.

We further study the properties of measures on  $\mathbf{B}$  corresponding to the unipotent traces. We show (see Theorem 4.1) that each such measure is ergodic (with respect to conjugations). Motivated by this connection we turn our attention to study of ergodic probability measures on  $\mathbf{B}$ .

At this point in order to simplify the exposition it is convenient to switch from  $\mathbf{GLB}$  to another distinguished infinite-dimensional matrix group  $\mathbf{GLU}$ .  $\mathbf{GLU}$  is the group of all almost uni-uppertriangular infinite matrices, i.e.

$$\mathbf{GLU} = \{[X_{ij}] \in \mathbf{GLB} : X_{ii} = 1 \text{ for large enough } i\}.$$

The whole theory for  $\mathbf{GLU}$  is very much parallel to that of  $\mathbf{GLB}$ , and we summarize it in the Appendix. For now we only need the fact that  $\mathbf{GLU}$  also has a distinguished class of extreme unipotent traces, enumerated by the very same sequence  $\alpha, \beta$ , moreover, under the identification  $\mathcal{A}(\mathbf{GLB})_n \simeq \mathbb{C}(\mathbf{GL}(n, q)) \simeq \mathcal{A}(\mathbf{GLU})_n$  unipotent traces of  $\mathbf{GLB}$  and  $\mathbf{GLU}$  are *the same* functions.



When we switch from **GLB** to **GLU** Borel subgroup **B** gets replaced by  $\mathbf{U} \subset \mathbf{GLU}$  which is the group of unipotent upper-triangular matrices. Note that, generally speaking, conjugations do not preserve  $\mathbf{U}$ . However, we can still define a conjugation-invariant measure  $\mu$  through the property  $\mu(X) = \mu(Y)$  for every  $X \subset U$ ,  $Y \subset U$  such that  $X = gYg^{-1}$ . We state and prove a partial result towards the classification theorem for ergodic conjugation-invariant measures on  $\mathbf{U}$ , see Conjecture 4.5 and Proposition 4.7. This theorem is a particular case of a general statement describing specializations of the algebra of symmetric functions  $\Lambda$  non-negative on Macdonald polynomials (our case corresponds to Hall-Littlewood polynomials), which is known as Kerov's conjecture, see [K03, Section II.9]. We also state a conjectural Law of Large numbers for ergodic measures, see Conjecture 4.5. One particular case of this conjecture was proved by Borodin [B1], [B2] who studied uniform measure on  $\mathbf{U}$ . Finally, in Theorem 4.6 we explain which measures in the above conjectural classification correspond to extreme unipotent traces.

**1.4. Construction of representations of **GLB**.** Our next topic is the construction of representations of **GLB** corresponding to the extreme unipotent traces of  $\mathcal{A}(\mathbf{GLB})$ . Of course, there exists an abstract general (Gelfand-Naimark-Segal) construction for the representation with given trace, but since by its definition **GLB** is a transformation group we seek for more explicit constructions based on its *natural* action in the infinite-dimensional vector space over  $\mathbb{F}_q$ .

In Section 5 we adopt the representation formalism of [O2] in a modified form and construct unitary representations  $T$  of  $(\mathbf{GLB} \times \mathbf{GLB})$  (in other words, we consider two-sided representations) in a Hilbert space  $H$  possessing a distinguished vector  $v$ , which is cyclic and invariant under the action of **GLB** diagonally embedded into  $(\mathbf{GLB} \times \mathbf{GLB})$ .  $v$  defines a spherical function  $\chi(a) = ((a, e)v, v)$ , and viewed as a function on  $\mathcal{A}(\mathbf{GLB})$  (or **GLB**) this function becomes our trace. There is a simple link between our constructions and factor representations (semifinite, in general). If we consider the restriction of  $T$  on the first component of  $\mathbf{GLB} \times \mathbf{GLB}$ , then we get a von Neumann factor representation.

In the most well-studied settings of asymptotic representation theory, e.g. for infinite symmetric group  $S(\infty)$  (see [Th64], [VK81], [O3], [Ok]) and real infinite-dimensional matrix groups such as  $U(\infty)$  (see [Vo], [O1], [O2]) distinguished vector  $v$  belongs to  $H$  and corresponding factor representation is of type  $II_1$ . However, in our settings, since our traces are defined only on  $\mathcal{A}(\mathbf{GLB})$  instead of the whole group **GLB**, we have to use generalized vectors (distributions)  $v$  and corresponding factor representations we get are of type  $II_\infty$ .

Similar situations appeared before in the investigation of at least two topics of classical representation theory of finite-dimensional groups. In the study of unitary representations of general Lie groups sometimes one is led to consider type  $II_\infty$  factor representation, see e.g. [Pu71], [Pu74] where characters and representations of simply connected solvable and more general Lie groups are studied. From the other side, in the theory of semisimple Lie groups generalized distinguished vectors show up<sup>1</sup> in harmonic analysis, i.e. when one tries to decompose highly-reducible representations. As a quick example, by the well-known theorem bi-regular representation of a finite group  $G$  equipped with a distinguished vector  $v = \delta$ -function at identity element of  $G$  — is the direct sum of irreducible spherical representations

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<sup>1</sup>Note, however, that all the representations are of type  $I$  in this theory.

$\pi^\lambda \otimes (\pi^\lambda)^*$  of Gelfand pair  $(G \times G, G)$  (here  $\lambda$  goes over all irreducible representations of  $G$ ). By Peter-Weyl theorem, the same decomposition is valid for a compact Lie group, however,  $\delta$ -function at identity is no longer a vector of  $L_2$  on the group, rather it is a generalized vector (distribution).

Coming back to the construction of representation of **GLB** recall that extreme unipotent traces of  $\mathcal{A}(\mathbf{GLB})$  are parameterized by two sequences of non-negative reals  $\{\alpha_i\}$  and  $\{\beta_i\}$ . We start from two distinguished simplest cases, where the corresponding factor representations are of type  $I$ . If  $\alpha_1 = 1$  with other parameters being zeros, then the corresponding representations is trivial 1-dimensional representation of **GLB**. On the contrary, if  $\beta_1 = 1$  with other parameters being zeros, then the corresponding von Neumann factor representation is of type  $I_\infty$  and the construction is related to classical *Steinberg representation* of  $\mathbb{GL}(n, q)$ , see [St], [Hu].

Further we concentrate on the case  $\beta_i = 0$  for all  $i$  and  $\sum_i \alpha_i = 1$ , where we get type  $II_\infty$  factor representations. To simplify the exposition let us stick to the case when only  $\alpha_1$  and  $\alpha_2$  are non-zero. Our construction starts with natural action of **GLB** on the grassmanian  $Gr(V)$ , which is the set of all subspaces of countable infinite-dimensional vector space  $V$  over  $\mathbb{F}_q$ . (If more than two  $\alpha_i$ s are non-zero, then  $Gr(V)$  gets replaced by an appropriate space of flags.) A well-known group-measure space (or crossed product) construction going back to the papers of F. J. Murray and J. von Neumann [MN], [N] associates a von Neumann factor to a free ergodic action of a group on a space equipped with measure. Of course, the action of **GLB** on  $Gr(V)$  is not free and, thus, modifications are necessary. The known solution here is to use *principal groupoid* of the equivalence relation spanned by the group action, in other words, we take the set  $Gr^2(V) \subset Gr(V) \times Gr(V)$  which is the graph of the equivalence relation. Groupoids and construction of the associated von Neumann algebras attracted a lot of attention in the literature, see [Mo] for a review, a somewhat simpler case when all the classes of equivalence relation are finite was first studied in [Kr] and [FM]. Groupoids in the context of asymptotic representation theory of symmetric groups first appeared in [VK81], where the construction was based on the action of  $S(\infty)$  on the set of words equipped with product measure.

When classes (orbits) are uncountable (which is the case for the action of **GLB** in  $Gr(V)$ ) the construction is more delicate, and even the definition of the correct measure on groupoid becomes complicated. General solutions do exist here, see [Ha], [Re], however, in our case the situation is simplified by the fact that  $Gr(V)$  can be decomposed into *Schubert cells*, each of which is a **B**-orbit. This lets us to start from a measure on symbols of Schubert cells (here we use the Bernoulli measure, similarly to the constructions of [VK81] for  $S(\infty)$ ) and produce the measures on  $Gr(V)$  and groupoid  $Gr^2(V)$  using it. We end up with quasiinvariant measures, which allow us to construct usual unitary representation of **GLB**  $\times$  **GLB** in the space of square-integrable functions on  $Gr^2(V)$ . As for the distinguished vector  $v$ , in the case of countable equivalence classes (as happens for  $S(\infty)$ ) the right choice is known to be the indicator function of the diagonal of groupoid, see [VK81], [FM]. In our case the diagonal has measure zero, so this choice is inappropriate. Because of that we have to use a generalized vector (distribution)  $v$  which is the integral along the diagonal (defined only for the continuous functions.)



One interesting aspect here is that the measure on  $Gr(V)$  we use is not  $\mathbf{GLB}$ -invariant and, moreover, one proves that there is no equivalent  $\sigma$ -finite  $\mathbf{GLB}$ -invariant measure. In classics this would imply that the resulting crossed product gives a factor of type  $III$ , see e.g. [Kr, Theorem 2.4], [SV, Theorem I.3.12], while we end up with type  $II_\infty$  factor representations of  $\mathbf{GLB}$ . An explanation here is the following: our trace on the operators of representation of  $\mathbf{GLB}$  can not be extended to the operators of the multiplication by the function, as opposed to the situations related to the variations of the crossed product construction.

Somewhat related question concerns cyclicity of the distinguished vector  $v$ . In general, there is no guarantee that  $\mathbf{GLB} \times \mathbf{GLB}$  orbit of our vector is dense, thus, we have to consider the representation in the cyclic hull of  $v$ . In [VK81], [O3, Section 5], [V11] the cyclicity question for the representations of infinite symmetric group  $S(\infty)$  was discussed; for  $\mathbf{GLB}$  this topic requires further investigations.

We are currently unable to construct representations of  $\mathbf{GLB}$  corresponding to other unipotent traces of  $\mathcal{A}(\mathbf{GLB})$ . Even the case  $\alpha_i = \beta_i = 0$  for all  $i$ , which for the infinite symmetric group  $S(\infty)$  corresponds to the biregular representation, is out of our reach at the moment.

In papers [V11], [V12] a new approach to the construction of spherical representations (or finite factor representations on the different language) related to infinite symmetric group  $S(\infty)$  was proposed. The authors hope that this approach might be extended to  $\mathbf{GLB}$ .

**1.5. Biregular representation of  $\mathbf{GLB}$ .** The final object of our interest is the (bi-)regular representation of  $\mathbf{GLB}$ . Since  $\mathbf{GLB}$  admits a unique Haar measure  $\mu_{\mathbf{GLB}}$  (normalized by the condition  $\mu_{\mathbf{GLB}}(\mathbf{B}) = 1$ ) there is a well-defined two-sided representation of  $\mathbf{GLB}$  in  $L_2(\mathbf{GLB}, \mu_{\mathbf{GLB}})$ . This representation equipped with the distinguished distribution— $\delta$ -function at unit element of the group—fits into the formalism of generalized spherical representations and corresponds to a certain trace of  $\mathcal{A}(\mathbf{GLB})$ . As in the classical harmonic analysis on finite or compact groups we are interested into the decomposition of this representation. In Section 6 we describe the decomposition of the trace of bi-regular representation of  $\mathbf{GLB}$  into a combination of extreme traces of  $\mathcal{A}(\mathbf{GLB})$ .

**1.6. Motivations and comments.** A well-known point of view is that the symmetric group  $S(n)$  can be viewed as  $\mathbb{GL}(n, q)$  over the field with one element, i.e. with  $q = 1$ . This agrees with similarities between the representation theory of  $\mathbb{GL}(n, q)$  and  $S(n)$ , so it is natural to expect some similarities for  $n = \infty$  as well.

The infinite symmetric group  $S(\infty)$  is usually defined as the inductive limit of finite symmetric group, equivalently,  $S(\infty)$  consists of all bijections of countable set, which permute only finitely many elements. A natural adaptation of this definition to  $\mathbb{GL}(n, q)$  is the following. Realize  $\mathbb{GL}(n, q)$  as a subgroup of  $\mathbb{GL}(n+1, q)$  acting in the space spanned by the first  $n$  coordinate vectors and fixing  $n+1$ st coordinate vector and consider the inductive limit of  $\mathbb{GL}(n, q)$  with respect to such embeddings. In this way we get the infinite-dimensional group  $\mathbb{GL}(\infty, q)$ . However, the representation theory of  $\mathbb{GL}(\infty, q)$  turns out to be not as rich as one could hope for. For instance, the set of extreme (indecomposable) characters of  $\mathbb{GL}(\infty, q)$  is countable (see [Th72], [Sk]) as opposed to the infinite symmetric group  $S(\infty)$  (see [Th64], [VK81], [KOO], [Ok]) or infinite-dimensional unitary group  $U(\infty)$  (see

[Vo], [VK82], [Bo], [OO], [BO], [Pe], [GP]) for which such sets comprise infinite-dimensional domains in  $\mathbb{R}^\infty$ . This leads one to seek for other  $n = \infty$  analogue of  $\mathbb{GL}(n, q)$ .

The key idea here is to change the embeddings. A new definition is hinted by the notions of *parabolic induction and restriction* well-known in the representation theory of  $\mathbb{GL}(n, q)$ , see [Gr], [Zel], [F]. This leads to *parabolic embeddings*  $i_n : \mathbb{C}(\mathbb{GL}(n, q)) \hookrightarrow \mathbb{C}(\mathbb{GL}(n+1, q))$  which are no longer induced by the group embeddings, see Section 2.2 for the formal definition. The inductive limit of  $\mathbb{C}(\mathbb{GL}(n, q))$  with respect to the embeddings  $i_n$  is our main hero — algebra  $\mathcal{A}(\mathbf{GLB})$ . A thorough analysis of the definitions leads to the realization of  $\mathcal{A}(\mathbf{GLB})$  as subalgebra of the algebra of the functions on a group, that's how the group  $\mathbf{GLB}$  first appears. Note that  $\mathbb{GL}(\infty, q)$  is a dense subgroup of  $\mathbf{GLB}$ , so another point of view might be to consider  $\mathbf{GLB}$  as a certain *completion* of discrete group  $\mathbb{GL}(\infty, q)$ .

The representation theory of  $\mathbf{GLB}$ , indeed, turns out to be similar to that of  $S(\infty)$ . First, the classification of traces of  $\mathcal{A}(\mathbf{GLB})$  and characters of  $S(\infty)$  are similar, sequences  $\{\alpha_i\}$  and  $\{\beta_i\}$  appear in both. The similarity is even more striking when one considers unipotent extreme traces of  $\mathcal{A}(\mathbf{GLB})$ . Their normalized versions are in one-to-one correspondence with extreme characters of  $S(\infty)$  and both families have similar properties, e.g. multiplicativity and the same coefficients of decomposition into irreducibles of restrictions to  $\mathcal{A}(\mathbf{GLB})_n (S(n))$ .

The constructions of the representations corresponding to characters are also similar, although some distinctions do exist (e.g. the distinguished vector becomes a distribution for  $\mathbf{GLB}$ ). More precisely, the realization of representations of  $\mathbf{GLB}$  with unipotent traces with non-zero parameters  $\alpha_i$  is related to the spaces of flags of subspaces, while corresponding representations of  $S(\infty)$  are related to its exact  $q = 1$  analogue which is the space of flags of subsets, see [VK81], [TV].

The above facts lets us claim that the group  $\mathbf{GLB}$  might be the right  $q$ -analogue of  $S(\infty)$  and  $n = \infty$  analogue of  $\mathbb{GL}(n, q)$  in the context of the asymptotic representation theory.

However, some of the similarities break down when we start considering representations with non-zero  $\beta_i$ . Representation of  $S(\infty)$  with single non-zero parameter  $\beta_1 = 1$  is the simple one-dimensional alternating representation, while the corresponding representation of  $\mathbf{GLB}$  is an infinite-dimensional one; this is parallel to the difference between alternating one-dimensional representation of  $S(n)$  and corresponding unipotent (principal) representation of  $\mathbb{GL}(n, q)$  which is the Steinberg representation of dimension  $q^{n(n-1)/2}$ .

More importantly, while the (bi-)regular representation of  $S(\infty)$  is irreducible and corresponds to zero parameters  $\alpha_i$  and  $\beta_i$ , the (bi-)regular representation of  $\mathbf{GLB}$  is reducible (as we explain in Section 6). The construction of the unipotent representation of  $\mathbf{GLB}$  corresponding to zero parameters at the moment remains unknown.

We intensively exploit the similarity between  $S(\infty)$  and  $\mathbf{GLB}$  in our methods. For instance, some theorems of the present article are based on the Ring Theorem, which originally was discovered in the study of  $S(\infty)$ , see [KV80], [K03] and also [GO1, Section 8.7]). Also Schur polynomials play an important role in the study of  $S(\infty)$ , while in the present paper we intensively use both Schur polynomials and their  $q$ -deformation — Hall-Littlewood polynomials.

In the classics, the representation theory of  $S(\infty)$  has numerous connections with the representation theory of  $U(\infty)$ , see [BO2] and references therein. A  $q$ -deformation of the character theory of  $U(\infty)$  related to the quantum groups was proposed in [G]. It is yet to discover whether the representation theory of **GLB** is somehow related to that  $q$ -deformation.

Finally, we remark that some results on the structure of **GLB** from the algebraic point of view can be also found in the literature, see [Ho], [GH] and references therein.

### 1.7. List of main notations and theorems.

#### Notations:

$S(n)$  symmetric group of rank  $n$

$\mathcal{H}_q(n)$  Iwahori–Hecke algebra of rank  $n$

$\mathbb{F}_q$  — finite field with  $q$  elements

$\mathbb{GL}(n, q)$  — group of all invertible  $n \times n$  matrices over  $\mathbb{F}_q$

$S(\infty)$ ,  $\mathcal{H}_q(\infty)$ ,  $\mathbb{GL}(\infty, q)$  — inductive limits of corresponding finite  $n$  objects

**GLB** — group of all almost uppertriangular infinite matrices over  $\mathbb{F}_q$

**B** — group of all uppertriangular infinite matrices over  $\mathbb{F}_q$

**B** $_n$  — subgroup of **B** of all matrices such that their top left  $n \times n$  corner is identity matrix

**BI** $_n$  — group of all uppertriangular  $n \times n$  matrices over  $\mathbb{F}_q$

**GLU** — group of all almost uni-uppertriangular infinite matrices over  $\mathbb{F}_q$

**U** — group of all uni-uppertriangular infinite matrices over  $\mathbb{F}_q$

$\mu_{\mathbf{GLB}}$  — Haar measure on **GLB** normalized by  $\mu_{\mathbf{GLB}}(\mathbf{B}) = 1$

$\mu_{\mathbf{GLU}}$  — Haar measure on **GLU** normalized by  $\mu_{\mathbf{GLU}}(\mathbf{U}) = 1$

$\mathcal{A}(\mathbf{GLB})$ ,  $\mathcal{A}(\mathbf{GLU})$  — algebra of all continuous functions with compact support (on the corresponding group) taking only finitely many values.

$\mathbb{Y}$  — set of all Young diagrams and also Young graph

$\mathcal{C}$  — set of all irreducible monic polynomials over  $\mathbb{F}_q$  other than  $x$  and 1

$\mathcal{C}_n$  — all degree  $n$  polynomials in  $\mathcal{C}$

$\mathcal{CY}_n$  — set of all maps from  $\mathcal{C}$  to  $\mathbb{Y}$  of degree  $n$

$\mathcal{CY}$  — disjoint union of sets  $\mathcal{CY}_n$ ,  $n = 1, 2, \dots$

$\pi^f$ ,  $\chi^f$  — irreducible complex representation of  $\mathbb{GL}(n, q)$  parameterized by

$f \in \mathcal{CY}_n$  and its conventional character

$\mathcal{CY}'$  — subset of  $\mathcal{CY}$  of maps  $f$  such that  $f("x - 1'") = \emptyset$ .

$\Lambda$  — algebra of symmetric (polynomial) functions in countably many variables

$h_n$ ,  $e_n$ ,  $p_n$  — complete homogeneous functions, elementary symmetric functions and Newton power sums, respectively

$s_\lambda$  — Schur function indexed by  $\lambda \in \mathbb{Y}$

$P_\lambda(\cdot; t)$ ,  $Q_\lambda(\cdot; t)$  — Hall-Littlewood  $P$  and  $Q$  functions with parameter  $t$ , indexed by  $\lambda \in \mathbb{Y}$

$\mathcal{S}p_{\alpha, \beta, \gamma}$  — homomorphism from  $\Lambda$  into  $\mathbb{C}$  indexed by two sequence of non-negative numbers  $\alpha = \{\alpha_i\}$ ,  $\beta = \{\beta_i\}$  and real number  $\gamma$  such that  $\sum_i (\alpha_i + \beta_i) \leq \gamma$ , and given by its values on power sums

$$\mathcal{S}p_{\alpha, \beta, \gamma}[p_1] = \gamma, \quad \mathcal{S}p_{\alpha, \beta, \gamma}[p_k] = \sum_i \alpha_i^k + (-1)^{k-1} \sum_i \beta_i^k, \quad k > 1$$

**Key theorems:**

*Proposition 2.6* on page 14 identifies  $\mathcal{A}(\mathbf{GLB})$  with the inductive limit of the group algebras  $\mathbb{C}(\mathbb{GL}(n, q))$ .

*Theorem 2.26* on page 19 provides the description of all extreme traces of  $\mathcal{A}(\mathbf{GLB})$ .

*Theorem 3.3* on page 21 gives the proof of multiplicativity of extreme unipotent traces of  $\mathcal{A}(\mathbf{GLB})$ .

*Theorems 3.4 and 3.5* on page 22 relate the values of extreme unipotent characters to specializations of Hall–Littlewood polynomials.

*Theorem 3.10* on page 28 identifies the restrictions of unipotent traces with extreme traces of Iwahori–Hecke algebra.

*Theorems 4.1 and 4.6* on pages 28 and 30 explain that each unipotent characters can be viewed as a probability measure.

*Conjecture 4.5* on page 29 gives the (conjectural) classification and law of large numbers for conjugation–invariant probability measures on infinite upper-triangular matrices.

*Theorem 5.11* on page 40 provides a construction for the representations of  $\mathbf{GLB}$  related to grassmanian.

*Theorem 5.12* on page 44 provides a construction for the representations of  $\mathbf{GLB}$  related to spaces of flags.

*Theorem 6.1* on page 45 describes the decomposition of the biregular representation of  $\mathbf{GLB}$ .

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## 2. THE GROUP $\mathbf{GLB}$ AND ITS SCHWARTZ–BRUHAT ALGEBRA $\mathcal{A}(\mathbf{GLB})$

**2.1. Basic definitions.** Let  $\mathbb{F}_q$  be the finite field with  $q$  elements and let  $\mathbb{GL}(n, q)$  denote the group of all invertible  $n \times n$  matrices over  $\mathbb{F}_q$ . For any matrix  $X$  we denote through  $X^{(n)}$  its top left  $n \times n$  corner.

**Definition 2.1.**  $\mathbf{GLB}$  is the group of all invertible almost upper-triangular matrices over  $\mathbb{F}_q$  in other words  $X = [X_{ij}]_{i,j=1}^{\infty}$  is an element of  $\mathbf{GLB}$  if there exists  $n$  such that:

- (1) The  $n \times n$  submatrix  $X^{(n)}$  is invertible,
- (2)  $X_{ij} = 0$  for all  $i$  such that  $i > j$  and  $i > n$ ,
- (3)  $X_{ii} \neq 0$  for  $i > n$ .

The group  $\mathbf{GLB}$  is an inductive limit of groups  $\mathbf{GLB}_n$ , where

$$\mathbf{GLB}_n = \{[X_{ij}] \in \mathbf{GLB} \mid X_{ij} = 0 \text{ if both } i > j \text{ and } i > n\},$$

in particular,  $\mathbf{GLB}_0 = \mathbf{B} \subset \mathbf{GLB}$  is the subgroup of all upper-triangular invertible matrices.

Each  $\mathbf{GLB}_n$  is a compact group (with topology of pointwise convergence of matrix elements).  $\mathbf{GLB}$  as an inductive limit of  $\mathbf{GLB}_n$  is a locally compact topological

group. Let  $\mu_{\mathbf{GLB}}$  denote the biinvariant Haar measure on  $\mathbf{GLB}$  normalized by the condition  $\mu_{\mathbf{GLB}}(\mathbf{B}) = 1$ .

The space  $L_1(\mathbf{GLB}, \mu_{\mathbf{GLB}})$  is a Banach involutive algebra with multiplication given by the convolution.

**Definition 2.2.**  $\mathcal{A}(\mathbf{GLB})$  is defined as the subalgebra of  $L_1(\mathbf{GLB}, \mu_{\mathbf{GLB}})$  formed by all locally constant functions with compact support. In other words, a function  $f(X)$  belongs to  $\mathcal{A}(\mathbf{GLB})$  if there exists  $n$  and a function  $f_n : \mathbb{GL}(n, q) \rightarrow \mathbb{C}$  such that:

$$f(X) = \begin{cases} f_n(X^{(n)}), & \text{if } X \in \mathbf{GLB}_n, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly,  $\mathcal{A}(\mathbf{GLB})$  is dense in  $L_1(\mathbf{GLB}, \mu_{\mathbf{GLB}})$ . Note that algebra  $\mathcal{A}(\mathbf{GLB})$  does not have a unit element.

**Definition 2.3.** A continuous function  $\chi : \mathcal{A}(\mathbf{GLB}) \rightarrow \mathbb{C}$  is a trace of  $\mathcal{A}(\mathbf{GLB})$  if

- (1)  $\chi$  is central, i.e.  $\chi(WU) = \chi(UW)$ ,
- (2)  $\chi$  is positive definite, i.e.  $\chi(W^*W) \geq 0$  for any  $W \in \mathcal{A}(\mathbf{GLB})$ ,

**Remark.** It is impossible to normalize the traces, i.e. for any  $a \in \mathcal{A}(\mathbf{GLB})$  there exists a trace  $\chi$  such that  $\chi(a) = 0$ .

A trace  $\chi$  is *indecomposable* if  $\chi = \alpha_1 \chi_1 + \alpha_2 \chi_2$  with  $\alpha_1 > 0$ ,  $\alpha_2 > 0$  implies that both  $\chi_1$  and  $\chi_2$  are multiples of  $\chi$ . In other words, indecomposable traces are elements of extreme rays of the convex cone of all traces.

**2.2.  $\mathcal{A}(\mathbf{GLB})$  as an inductive limit.** For any matrix  $g \in \mathbb{GL}(n, q)$  let  $I_g^{\mathbf{GLB}} \in \mathcal{A}(\mathbf{GLB})$  denote the function

$$I_g^{\mathbf{GLB}}(X) = \begin{cases} 1, & \text{if } X \in \mathbf{GLB}_n \text{ and } X^{(n)} = g, \\ 0, & \text{otherwise.} \end{cases}$$

Let  $e(n)$  denote the identity element of  $\mathbb{GL}(n, q)$ . Then by the definition

$$I_g^{\mathbf{GLB}}(X) = I_{e(n)}^{\mathbf{GLB}}(Xg^{-1}) = g \cdot I_{e(n)}^{\mathbf{GLB}}.$$

**Definition 2.4.**  $\mathcal{A}(\mathbf{GLB})_n$  is defined as the linear span of  $I_g^{\mathbf{GLB}}$ ,  $g \in \mathbb{GL}(n, q)$ . Put it otherwise,  $\mathcal{A}(\mathbf{GLB})_n$  consists of functions from  $\mathcal{A}(\mathbf{GLB})$  with support in  $\mathbf{GLB}_n$  and depending only on the restriction of operator  $g \in \mathbf{GLB}_n$  on  $V_n \subset V_\infty$ .

The following proposition is straightforward

**Proposition 2.5.**  $\mathcal{A}(\mathbf{GLB})_n$  is a subalgebra of  $\mathcal{A}(\mathbf{GLB})$  isomorphic to the conventional group algebra  $\mathbb{C}(\mathbb{GL}(n, q))$ . If  $e_g$  denotes the natural basis of  $\mathbb{C}(\mathbb{GL}(n, q))$  then the isomorphism is given by  $e_g \rightarrow (q-1)^n q^{n(n-1)/2} I_g^{\mathbf{GLB}}$

Observe that  $\mathcal{A}(\mathbf{GLB})_n \subset \mathcal{A}(\mathbf{GLB})_{n+1}$ . In the basis  $I_g^{\mathbf{GLB}}$  this inclusion is given by

$$i_n : I_g^{\mathbf{GLB}} \rightarrow \sum_{h \in \text{Ext}^{\mathbf{GLB}}(g)} I_h^{\mathbf{GLB}},$$

where for  $g \in \mathbb{GL}(n, q)$  we have

$$Ext^{\mathbf{GLB}}(g) = \left\{ [h_{ij}] \in \mathbb{GL}(n+1, q) \mid \begin{array}{l} h^{(n)} = g \text{ and } h_{n,1} = h_{n,2} = \dots = h_{n,n-1} = 0 \end{array} \right\}.$$

Summarizing the discussion of this section we get the following statement.

**Proposition 2.6.** *The algebra  $\mathcal{A}(\mathbf{GLB})$  can be identified with the inductive limit of algebras  $\mathcal{A}(\mathbf{GLB})_n$ :*

$$\mathcal{A}(\mathbf{GLB}) = \varinjlim_{n \rightarrow \infty} \mathcal{A}(\mathbf{GLB})_n = \bigcup_n \mathcal{A}(\mathbf{GLB})_n.$$

For every  $n$  the algebra  $\mathcal{A}(\mathbf{GLB})_n$  is isomorphic to the group algebra  $\mathbb{C}(\mathbb{GL}(n, q))$

Thus,  $\mathcal{A}(\mathbf{GLB})$  is a locally semisimple algebra.

**2.3. Facts from representation theory of  $\mathbb{GL}(n, q)$ .** Let us fix the notations and recall some basic facts from the representation theory of the group  $\mathbb{GL}(n, q)$  which immediately translate into the statements for the representations and traces of algebra  $\mathcal{A}(\mathbf{GLB})_n$ . To a large extent we adopt the notations of the book [M].

A Young diagram  $\lambda$  is a finite collection of boxes arranged in rows with nonincreasing row lengths  $\lambda_i$ . The total number of boxes in  $\lambda$  is denoted by  $|\lambda|$ . Let  $\mathbb{Y}$  denote the set of all Young diagrams. We agree that the empty set  $\emptyset \in \mathbb{Y}$  and  $|\emptyset| = 0$ .  $\mathbb{Y}_n \subset \mathbb{Y}$  stays for the set of all Young diagrams with  $n$  boxes. Also for the Young diagram  $\lambda$  its transpose diagram is denoted  $\lambda'$ ; the row lengths of  $\lambda$  coincide with column lengths of  $\lambda'$ . For a box  $\square \in \lambda$  its hook length  $h(\square)$  is one plus number of the boxes below  $\square$  (in the same column) plus number of the boxes to the right from  $\square$  (in the same row).

For  $d > 1$  let  $\mathcal{C}_d$  denote the set of all monic irreducible polynomials of degree  $d$  over  $\mathbb{F}_q$ . Let  $\mathcal{C}_1$  be the set of all linear polynomials  $x - a$ ,  $a \in \mathbb{F}_q^*$ , i.e. we exclude the polynomial  $x$ . Clearly,  $|\mathcal{C}_1| = q - 1$ . Let  $\mathcal{C} = \bigcup_{d=1}^{\infty} \mathcal{C}_d$ .

**Definition 2.7.** A family of Young diagrams over the set  $\mathcal{C}$  is a map

$$\phi : \mathcal{C} \rightarrow \mathbb{Y},$$

such that

$$|\phi| := \sum_d \sum_{c \in \mathcal{C}_d} d |\phi(c)| < \infty.$$

We call  $|\phi|$  the degree of  $\phi$ .

**Definition 2.8.**  $\mathcal{CY}_d$  is the set of all families of degree  $d$  and  $\mathcal{CY} = \bigcup_{d=0}^{\infty} \mathcal{CY}_d$ .

**Theorem 2.9.** Irreducible representations of  $\mathbb{GL}(n, q)$  are parameterized by elements of  $\mathcal{CY}_d$ . The dimension of irreducible representation parameterized by  $\phi \in \mathcal{CY}_d$  is given by the  $q$ -analogue of the hook formula

$$\dim_q(\phi) = (q^n - 1) \dots (q - 1) \prod_{d \geq 1} \prod_{c \in \mathcal{C}_d} \frac{q^{dn(\phi(c)')}}{\prod_{\square \in \phi(c)} (q^{dh(\square)} - 1)}.$$



For the proof, construction of the representations and their characters see [Gr], [Zel], [M]. For  $f \in \mathcal{CY}_d$  let  $\pi^f$  denote the corresponding irreducible representation,  $H(\pi^f)$  the space of this representation, and let  $\chi^f(\cdot)$  be its conventional character (i.e. matrix trace of  $\pi^f(\cdot)$ ).

**Corollary 2.10.** *The set of all traces of  $\mathcal{A}(\mathbf{GLB})_n$  is a simplicial cone spanned by traces  $\chi^f$ . In other words, if  $\chi^n$  is a trace of  $\mathcal{A}(\mathbf{GLB})_n$ , then there exist unique nonnegative coefficients  $c(f)$  such that*

$$\chi^n(\cdot) = \sum_{f \in \mathcal{CY}_n} c(f) \chi^f(\cdot).$$

*Proof.*  $\mathcal{A}(\mathbf{GLB})_n$  is isomorphic to the conventional group algebra of  $\mathbb{GL}(n, q)$ . Under this correspondence a trace of  $\mathcal{A}(\mathbf{GLB})_n$  turns into the character of  $\mathbb{GL}(n, q)$ , i.e. central (class) positive-definite function on the group. It is well-known that characters of a finite group form a cone spanned by the characters (matrix traces) of the irreducible representations.  $\square$

Next we describe the interrelations between traces and inclusions  $i_n$ .

Embed  $\mathbb{GL}(n-1, q) \times \mathbb{GL}(1, q)$  into  $\mathbb{GL}(n, q)$  as the subgroup of block diagonal matrices. Consider the subgroup  $U_n^n \subset \mathbb{GL}(n, q)$  consisting of unipotent upper triangular matrices  $[u_{ij}]$  such that  $u_{ij}$  is non-zero only for  $j = n$  (and  $u_{nn} = 1$ ). Note that  $\mathbb{GL}(n-1, q) \times \mathbb{GL}(1, q)$  normalizes  $U_n^n$ .

**Theorem 2.11.** *Suppose that  $f \in \mathcal{CY}$ . Let  $\hat{H}(\pi^f)$  denote the subspace of  $U_n^n$ -invariant vectors in  $H(\pi^f)$ . And let  $\hat{\pi}^f$  denote the representation  $\mathbb{GL}(n-1, q) \times \mathbb{GL}(1, q)$  in this subspace. Let  $\{f_i\}$  be all families in  $\mathcal{CY}_{n-1}$  for which there exist  $y_i \in \mathcal{C}_1$  such that*

- (1)  $f_i(x) = f(x)$  for  $x \neq y_i$ ,
- (2) The difference of the Young diagrams  $f(y_i) \setminus f_i(y_i)$  is a single box.

Finally, let  $f \setminus f_i$  denote the family from  $\mathcal{CY}_1$  such that  $(f \setminus f_i)(y_i)$  is the one box diagram.

We have

$$\hat{\pi}^f = \bigoplus_i \pi^{f_i} \otimes \pi^{f \setminus f_i},$$

*Proof.* See e.g. [Zel, Chapter III].  $\square$

**2.4. Structure of  $\mathcal{A}(\mathbf{GLB})$ .** We need to introduce some notations to state an important corollary of Theorem 2.11.

**Definition 2.12.** *For two families  $f \in \mathcal{CY}_n$  and  $g \in \mathcal{CY}_{n-1}$  we say that  $g$  precedes  $f$  and write  $g \prec_{\mathbf{GLB}} f$  if*

- (1)  $f("x-1") \setminus g("x-1")$  is one box
- (2)  $f(u) = g(u)$  for all  $u \neq "x-1"$ .

(Here " $x-1$ "  $\in \mathcal{C}_1$  stays for the corresponding irreducible polynomial.)

**Theorem 2.13** (Branching rule). *Let  $\pi^f$  be the irreducible representation of algebra  $\mathcal{A}(\mathbf{GLB})_n$  (equivalently, of the group  $\mathbb{GL}(n, q)$ ) parameterized by  $f \in \mathcal{CY}_n$  and let  $\chi^f$  be its conventional character (i.e. matrix trace). The restrictions of  $\pi^f$  and  $\chi^f$  to the subalgebra  $\mathcal{A}(\mathbf{GLB})_{n-1}$  admit the following decomposition:*

$$\chi^f|_{\mathcal{A}(\mathbf{GLB})_{n-1}} = \sum_{g \prec_{\mathbf{GLB}} f} \chi^g,$$

equivalently,

$$\pi^f|_{\mathcal{A}(\mathbf{GLB})_{n-1}} = \mathcal{N} \oplus \bigoplus_{g \prec_{\mathbf{GLB}} f} \chi^g,$$

where  $\mathcal{N}$  is a zero representations of  $\mathcal{A}(\mathbf{GLB})_{n-1}$  of dimension  $\dim(f) - \sum_{g \prec_{\mathbf{GLB}} f} \dim(g)$ .

**Remark 1.** By zero representation we mean the action of  $\mathcal{A}(\mathbf{GLB})_{n-1}$  by the identical zero in a vector space of arbitrary dimension.

**Remark 2.** Theorem 2.13 implies, in particular, that the restriction of  $\pi^f$  to  $\mathcal{A}(\mathbf{GLB})_{n-1}$  is multiplicity free. This property was mentioned by various authors, the first proof was given by A. Zelevinski [Zel, Chapter III] using the Hopf algebras approach. Now there exist simple direct proofs of this fact, see [Go], [AG].

**Remark 2.** As opposed to the situation with parabolic embeddings, the restrictions of irreducible representations of  $\mathbb{GL}(n, q)$  to the naturally embedded subgroup  $\mathbb{GL}(n-1, q)$  are *not* multiplicity free, see [Th71], [Zel, Chapter III, Section 13].

*Proof.* This follows from Theorem 2.11 and we use the notations of that theorem. Indeed, the summation in the definition of parabolic embedding  $i_{n-1}$  introduces averaging over  $U_n^n$  and over  $\mathbb{GL}(1, q)$ . Therefore, the parabolic embedding translates into the projection on  $\mathbb{GL}(1, q)$ -invariants in  $\hat{\pi}^f$ .  $\square$

Now the structure of locally semisimple algebra can be encoded via its *Bratteli diagram* [Br], [VK87], [K03].

**Definition 2.14.**  $B(\mathbf{GLB})$  is a graded graph supplemented with additional numbers, labels of the vertices. The set  $B(\mathbf{GLB})_n$  of vertices at level  $n$  is  $\mathcal{CY}_n$ . The label  $l(f)$  of the vertex  $f \in B(\mathbf{GLB})_n$  is the dimension of the irreducible representation of  $\mathbb{GL}(n, q)$  parameterized by  $f$ , the formula for its computation is given in Theorem 2.9. An edge joins vertex  $f$  and vertex  $g$  is and only if  $g \prec_{\mathbf{GLB}} f$ .

Theorem 2.13 implies that

**Proposition 2.15.**  $B(\mathbf{GLB})$  is the Bratteli diagram of algebra  $\mathcal{A}(\mathbf{GLB})$ .

For convenience of the reader we recall in our settings the general procedure for the reconstruction of the involutive algebra by its Bratteli diagram.

By the well-known theorem algebra  $\mathcal{A}(\mathbf{GLB})_n$  is isomorphic to the direct sum of matrix algebras of ranks equal to the dimensions of its irreducible representations. Therefore,

$$(2.1) \quad \mathcal{A}(\mathbf{GLB})_n = \bigoplus_{f \in B(\mathbf{GLB})_n} \text{Mat}(l(f), l(f))$$

The inclusions  $i_n : \mathcal{A}(\mathbf{GLB})_n \hookrightarrow \mathcal{A}(\mathbf{GLB})_{n+1}$  can be reconstructed as follows. For every  $f \in B(\mathbf{GLB})_n$  fix the embedding

$$\bigoplus_{g \prec_{\mathbf{GLB}} f} \text{Mat}(l(g), l(g)) \hookrightarrow \text{Mat}(l(f), l(f))$$

as block-diagonal matrices. Note that we need the inequality

$$\sum_{g \prec_{\mathbf{GLB}} f} l(g) \leq l(f)$$

to be satisfied. Let  $i_{g,f}$  denote the above embedding considered as a map from the matrix algebra corresponding to  $g$  to the matrix algebra corresponding to  $f$  viewed as a subalgebra of  $\mathcal{A}(\mathbf{GLB})_{n+1}$ .

Now for  $a = \sum_{f \in B(\mathbf{GLB})_n} m_f$ , with  $m_f \in \text{Mat}(l(f), l(f))$  in (2.1), we set

$$i_n(a) = \sum_{f \in B(\mathbf{GLB})_{n+1}} \sum_{g \prec f} i_{g,f} m_g.$$

Algebra  $\mathcal{A}(\mathbf{GLB})$  is reconstructed as the inductive limit of  $\mathcal{A}(\mathbf{GLB})_n$ .

**2.5. Some subalgebras of  $\mathcal{A}(\mathbf{GLB})$ .** Let  $\mathbf{B}_n \subset \mathbb{GL}(n, q)$  be the (Borel) subgroup of all upper-triangular matrices. We call an element

$$a = \sum_{g \in \mathbb{GL}(n, q)} c(g) e_g \in \mathbb{C}(\mathbb{GL}(n, q))$$

$B_n$ -biinvariant if  $c(g) = c(b_1 g b_2)$  for any  $g \in \mathbb{GL}(n, q)$  and  $b_1, b_2 \in \mathbf{B}_n$ . Put it otherwise,  $B_n$ -biinvariant element is a linear combination of characteristic functions of double cosets  $B_n g B_n$ .

**Definition 2.16.** *The Iwahori–Hecke algebra  $\mathcal{H}_q(n)$  is defined as the algebra of  $B_n$ -biinvariant elements in  $\mathbb{C}(\mathbb{GL}(n, q))$ .*

The following proposition describes the structure of  $\mathcal{H}_q(n)$ .

**Proposition 2.17.** *The algebra  $\mathcal{H}_q(n)$  has dimension  $n!$  and has a linear basis  $s_\omega$  enumerated by permutation matrices  $\omega$ :*

$$s_\omega = \frac{1}{|\mathbf{B}_n|} \sum_{g \in \mathbf{B}_n \omega \mathbf{B}_n} e_g.$$

As an algebra  $\mathcal{H}_q(n)$  is generated by  $n - 1$  elements  $s_{(i, i+1)}$  (where  $(i, i+1)$  is elementary transposition permuting  $i$  and  $i+1$ ) subject to relations

- (1)  $s_{(i, i+1)} s_{(j, j+1)} = s_{(j, j+1)} s_{(i, i+1)}, \quad |i - j| > 1,$
- (2)  $s_{(k, k+1)} s_{(k+1, k+2)} s_{(k, k+1)} = s_{(k+1, k+2)} s_{(k, k+1)} s_{(k+1, k+2)},$
- (3)  $s_{(k, k+1)}^2 = (q - 1) s_{(k, k+1)} + q s_e,$

where  $e$  is identical permutation.  $s_e$  is the unit element in  $\mathcal{H}_q(n)$ .

*Proof.* See [I], [Bou]. □

Let us embed  $\mathcal{H}_q(n)$  into  $\mathcal{H}_q(n+1)$  as a subalgebra spanned by first  $n - 1$  out of  $n$  generators.

**Definition 2.18.** *The infinite-dimensional Iwahori–Hecke algebra  $\mathcal{H}_q(\infty)$  is defined as the inductive limit of  $\mathcal{H}_q(n)$ :*

$$\mathcal{H}_q(\infty) = \varinjlim_{n \rightarrow \infty} \mathcal{H}_q(n) = \bigcup_n \mathcal{H}_q(n).$$

Note that  $\mathcal{H}_q(\infty)$  is generated by countably many generators  $s_{(i, i+1)}$  subject to the same relations as in Proposition 2.17.

Through the identification  $\mathbb{C}(\mathbb{GL}(n, q)) \simeq \mathcal{A}(\mathbf{GLB})_n$  we can view  $\mathcal{H}_q(n)$  as a subalgebra of  $\mathcal{A}(\mathbf{GLB})_n$ . The following proposition is straightforward.

**Proposition 2.19.** *The restriction of the embedding  $i_n : \mathcal{A}(\mathbf{GLB})_n \rightarrow \mathcal{A}(\mathbf{GLB})_{n+1}$  on the subalgebra  $\mathcal{H}_q(n)$  coincides with above embedding  $\mathcal{H}_q(n) \rightarrow \mathcal{H}_q(n+1)$ , therefore,  $\mathcal{H}_q(\infty) \subset \mathcal{A}(\mathbf{GLB})$ .  $\mathcal{H}_q(\infty)$  coincides with subalgebra of  $\mathbf{B}$ -biinvariant functions in  $\mathcal{A}(\mathbf{GLB}) \subset L_1(\mathbf{GLB}, \mu_{\mathbf{GLB}})$ .*

We want to define yet another important subalgebra of  $\mathcal{A}(\mathbf{GLB})$ . Let  $I_{\mathbf{B}} \in \mathcal{A}(\mathbf{GLB})$  denote the indicator function of  $\mathbf{B}$  in the realization of  $\mathcal{A}(\mathbf{GLB})$  as a subalgebra of  $L_1(\mathbf{GLB}, \mu_{\mathbf{GLB}})$ .

**Definition 2.20.** *The unipotent subalgebra  $\mathcal{A}(\mathbf{Uni})$  is defined as a two-sided ideal in  $\mathcal{A}(\mathbf{GLB})$  generated by  $I_{\mathbf{B}}$ .*

Our definitions imply that  $\mathcal{H}_q(\infty) \subset \mathcal{A}(\mathbf{Uni}) \subset \mathcal{A}(\mathbf{GLB})$ .

We also note that the Bratteli diagram of  $\mathcal{A}(\mathbf{GLB})$  described in the previous section is a disjoint union of countably many copies of the *Young graph* with shifted gradings and different labels of the vertices. Therefore,  $\mathcal{A}(\mathbf{GLB})$  is the direct sum of ideals corresponding to the connected components of its Bratteli diagram. We remark that  $\mathcal{A}(\mathbf{Uni})$  is precisely the component consisting of families  $f \in \mathcal{CY}$  such that  $f(u) = \emptyset$ , unless  $u = "x-1"$ , see also Proposition 3.1 for a related fact.

**2.6. Classification of traces of  $\mathcal{A}(\mathbf{GLB})$ .** Although, we are not going to use it directly, but the following abstract statement holds:

**Proposition 2.21.** *The description of traces of a locally semisimple algebra depends solely on its Bratteli diagram without labels. In other words, if  $\mathcal{X}$  and  $\mathcal{Y}$  are two locally semisimple algebras, whose Bratteli diagrams have the same sets of vertices and edges but, perhaps, different labels of vertices, then there is a canonical correspondence between their traces.*

*Sketch of the proof.* This follows from the identification of traces with *harmonic functions* or *coherent systems* on the Bratteli diagram of the algebra, see [VK87], [VK90], [K03] for more details. The key idea here is that branching of traces does not depend on labels, for  $\mathbf{GLB}$  this can be seen in Theorem 2.13.  $\square$

In order to state the classification theorem for traces of  $\mathcal{A}(\mathbf{GLB})$  we need to introduce some additional notations.

Let  $f \in \mathcal{CY}$  be a family of Young diagrams. We call the set

$$\{x \in \mathcal{C} \mid f(x) \neq \emptyset\}$$

the support of  $f$  and denote it  $\text{supp}(f)$ . If  $f$  and  $g$  are two families of Young diagrams with disjoint supports, then  $f + g$  stays for the following family:

$$(f + g)(x) = \begin{cases} f(x), & x \in \text{supp}(f), \\ g(x), & x \in \text{supp}(g), \\ \emptyset, & \text{otherwise.} \end{cases}$$

This operation corresponds to the *parabolic induction* of representations of  $\mathbf{GL}(n, q)$  (see e.g. [Gr], [Zel, Chapter III] and [M, Section IV.3])

Let  $\Lambda$  be the algebra of symmetric functions in variables  $x_1, x_2, \dots$  (see e.g. [M] for all the definitions). We intensively use various generators of this algebra, namely, elementary symmetric functions  $e_n$ , complete symmetric functions  $h_n$  and power sums  $p_k$ :

$$p_k = \sum_i x_i^k.$$

We also use *Schur symmetric functions*  $s_\lambda$ ,  $\lambda \in \mathbb{Y}$  which form a linear basis in  $\Lambda$ .

A *specialization*  $\Phi$  of  $\Lambda$  is an algebra homomorphism:

$$\Phi : \Lambda \rightarrow \mathbb{C}.$$

Note that any specialization of  $\Lambda$  is uniquely defined by its values on  $p_k$ . In what follows we write the arguments of specializations in square brackets  $\Phi[\cdot]$ .

Let  $\alpha = \{\alpha_i\}$  and  $\beta = \{\beta_i\}$ ,  $i = 1, 2, 3, \dots$  be two weakly decreasing sequences of non-negative real numbers such that

$$(2.2) \quad \sum_{i=1}^{\infty} (\alpha_i + \beta_i) \leq \gamma < \infty.$$

**Definition 2.22.** For any two sequences  $\alpha$  and  $\beta$  of non-negative reals and number  $\gamma$  satisfying (2.2) we define the specialization  $Sp_{\alpha, \beta, \gamma}$  through its values on the generators  $p_k$  of  $\Lambda$

$$Sp_{\alpha, \beta, \gamma}[p_1] = \gamma, \quad Sp_{\alpha, \beta, \gamma}[p_k] = \sum_i \alpha_i^k + (-1)^{k-1} \sum_i \beta_i^k.$$

**Remark.** Note that if  $\beta_i = 0$  and  $\sum_i \alpha_i = \gamma$ , then the specialization  $Sp_{\alpha, \beta, \gamma}$  boils down to the substitution of numbers  $\alpha_i$  in place of formal variables  $x_i$ .

**Definition 2.23.**  $\mathcal{CY}' \subset \mathcal{CY}$  is the set of families  $f$  of Young diagrams such that  $f(\text{"}x-1\text{"}) = \emptyset$ .

**Definition 2.24.**  $\Omega(\mathbf{GLB})$  is defined as the set of triplets  $(\alpha, \beta, f)$ , where  $\alpha = \{\alpha_i\}$  and  $\beta = \{\beta_i\}$ ,  $i = 1, 2, 3, \dots$  are two weakly decreasing sequences of non-negative real numbers satisfying (2.2) for  $\gamma = 1$  and  $f \in \mathcal{CY}'$ .

**Definition 2.25.** For  $\omega \in \Omega(\mathbf{GLB})$  we define a trace  $\chi^\omega$  of  $\mathcal{A}(\mathbf{GLB})$  as follows. For  $g \in \mathbf{GL}(n, q)$  we have  $\chi^\omega(I_g^{\mathbf{GLB}}) = 0$  if  $n < |f|$ , otherwise,

$$(2.3) \quad \chi^\omega(I_g^{\mathbf{GLB}}) = \sum_{\lambda \in \mathbb{Y}_{n-|f|}} \chi^{f+E_1(\lambda)}(I_g^{\mathbf{GLB}}) Sp_{\alpha, \beta, 1}[s_\lambda],$$

where  $E_1(\lambda)$  is a function from  $\mathcal{CY}_m$  taking value  $\lambda$  in  $\text{"}x-1\text{"}$  and taking value  $\emptyset$  in all other points.  $\chi^{f+E_1(\lambda)}$ , as and above, stays for the matrix trace of the irreducible representation of  $\mathcal{A}(\mathbf{GLB})_n$  ( $\mathbf{GL}(n, q)$ ) indexed by  $f + E_1(\lambda)$ .

**Theorem 2.26** (Classification theorem for finite traces of  $\mathcal{A}(\mathbf{GLB})$ ). The extreme rays of the set of traces of  $\mathcal{A}(\mathbf{GLB})$  are parameterized by elements of  $\Omega(\mathbf{GLB})$ . For  $\omega = (\alpha, \beta, f) \in \Omega(\mathbf{GLB})$  the corresponding ray is  $\mathbb{R}_+ \chi^\omega(\cdot)$ .

*Proof.* For a family  $f \in \mathcal{CY}'$  let  $\mathcal{CY}^{(f)} \subset \mathcal{CY}$  denote the set of families  $h \in \mathcal{CY}$  such that  $h(u) = f(u)$  for all  $u \in \mathcal{C} \setminus \{\text{"}x-1\text{"}\}$ .

Moreover, for a family  $f \in \mathcal{CY}'$  let  $\Upsilon^f$  denote the convex cone of traces  $\chi$  of  $\mathcal{A}(\mathbf{GLB})$  such that for  $n < |f|$  the restriction  $\chi|_{\mathcal{A}(\mathbf{GLB})_{n-1}}$  vanishes and for  $n \geq |f|$  in the decomposition (see Corollary 2.10)

$$\chi|_{\mathcal{A}(\mathbf{GLB})_n} = \sum_{h \in \mathcal{CY}_n} c(h) \chi^h(\cdot).$$

$c(h) = 0$  unless  $h \in \mathcal{CY}^{(f)}$ . Let  $\Upsilon^\emptyset$  denote the set  $\Upsilon^f$  for  $f$  being the empty family. We claim that for any  $f \in \mathcal{CY}'$  the convex cone  $\Upsilon^f$  is affine isomorphic to  $\Upsilon^\emptyset$ . The isomorphism

$$\Phi^f : \Upsilon^\emptyset \rightarrow \Upsilon^f$$

is given by the following formula. If  $\chi \in \Upsilon^\emptyset$  is such that

$$\chi|_{\mathcal{A}(\mathbf{GLB})_n} = \sum_{h \in \mathcal{CY}_n \cap \mathcal{CY}(f)} c(h) \chi^h(\cdot),$$

then

$$\Phi^f(\chi)|_{\mathcal{A}(\mathbf{GLB})_{n+|f|}} = \sum_{h \in \mathcal{CY}_n \cap \mathcal{CY}(f)} c(h) \chi^{h+f}(\cdot).$$

Theorem 2.13 implies that the branching of traces from  $\Upsilon^\emptyset$  with respect to restriction on subalgebras  $\mathcal{A}(\mathbf{GLB})_n$  is the same as branching of the characters of symmetric groups, cf. [Sa], [K03]. Therefore,  $\Upsilon^\emptyset$  is isomorphic to the set of characters of the infinite symmetric group  $S(\infty)$ , see [VK90], [K03]. The latter characters were classified by Thoma [Th64], see also [VK81]. Thoma's theorem implies that the extreme rays of  $\Upsilon^\emptyset$  are parameterized by pairs  $\alpha = \{\alpha_i\}$  and  $\beta = \{\beta_i\}$ ,  $i = 1, 2, 3, \dots$  of weakly decreasing sequences of non-negative real numbers satisfying (2.2) with  $\gamma = 1$ . The ray corresponding to a pair  $(\alpha, \beta)$  is spanned by the character  $\chi^{\alpha, \beta}$  such that for  $g \in \mathbb{GL}(n, q)$  we have

$$\chi^{\alpha, \beta}(I_g^{\mathbf{GLB}}) = \sum_{\lambda \in \mathbb{Y}_n} \chi^\lambda(I_g^{\mathbf{GLB}}) \mathcal{S}p_{\alpha, \beta, 1}[s_\lambda],$$

We conclude that for  $f \in \mathcal{CY}'$  the extreme rays of  $\Upsilon^f$  are parameterized by pairs  $(\alpha, \beta)$  and are given by the formula (2.3) for the triplet  $(\alpha, \beta, f)$ .

It remains to prove that every extreme ray of the set of traces of  $\mathcal{A}(\mathbf{GLB})$  is an extreme ray of one of the sets  $\Upsilon^f$ . Indeed, let  $\chi$  be a trace of  $\mathcal{A}(\mathbf{GLB})$ . We claim that there exists a unique decomposition of  $\chi$  into the sum

$$\chi = \sum_{f \in \mathcal{CY}'} \chi^{(f)}, \quad \chi^{(f)} \in \Upsilon^f$$

To prove the claim consider the restrictions  $\chi^{(f)}|_{\mathcal{A}(\mathbf{GLB})_n}$  for which the existence and uniqueness of such decomposition is immediate. This finishes the proof.  $\square$

### 3. UNIPOTENT TRACES AND THEIR VALUES

Recall that an irreducible representation of  $\mathbb{GL}(n, q)$  is said to be *unipotent* (see e.g. [St], [J2]) if it contains a non-zero  $\mathbf{B}_n$ -invariant vector. (Here  $\mathbf{B}_n \subset \mathbb{GL}(n, q)$  is the subgroup of upper-triangular matrices.) In the above parameterization of irreducible representation of  $\mathbb{GL}(n, q)$  by the families of Young diagrams, unipotent representations  $\pi^f$  are precisely those for which  $f(p) = \emptyset$  if  $p \neq "x-1"$ .

**Proposition 3.1.** *Let  $\chi^\omega$ ,  $\omega \in \Omega(\mathbf{GLB})$  be an indecomposable trace of  $\mathcal{A}(\mathbf{GLB})$ . The following conditions are equivalent:*

- (1) *For every  $n$  the restriction of  $\chi^\omega$  to  $\mathcal{A}(\mathbf{GLB})_n$  is a linear combination of matrix traces of irreducible unipotent representations of  $\mathbb{GL}(n, q)$ ,*
- (2) *Restriction of  $\chi^\omega$  on  $\mathcal{H}_q(\infty)$  is non-zero,*
- (3) *Restriction of  $\chi^\omega$  on  $\mathcal{A}(\mathbf{Uni})$  is non-zero,*
- (4)  *$\omega = (\alpha, \beta, f)$  with  $f \equiv \emptyset$ .*

*Proof.* Equivalence of properties (1) and (4) is a corollary of Theorem 2.26. Next, normalized indicator function of the Borel subgroup  $\mathbf{B}_n \subset \mathbb{GL}(n, q)$  is a unit element of  $\mathcal{H}_q(n)$ . In the same time in a unipotent representation of  $\mathbb{GL}(n, q)$  it



acts as a projection on the set of  $\mathbf{B}_n$ -invariant vectors, while in any other representation it acts as zero. Therefore, the value of the matrix trace of a unipotent representation of  $\mathbf{GL}(n, q)$  on this indicator function is positive, and (1) implies (2). Since  $\mathcal{H}_q(\infty) \subset \mathcal{A}(\mathbf{Uni})$ , the property (3) follows from (2). Finally, since  $\mathcal{A}(\mathbf{Uni})$  is spanned by the indicator function of  $\mathbf{B}$  and this indicator function vanishes in any non-unipotent representation, every element of  $\mathcal{A}(\mathbf{Uni})$  acts as zero in any non-unipotent representation. Therefore, the value of a non-unipotent character of  $\mathbf{GL}(n, q)$  on an element of  $\mathcal{A}(\mathbf{Uni})$  is zero and (1) follows from (3).  $\square$

**Definition 3.2.** *An indecomposable trace of  $\mathcal{A}(\mathbf{GLB})$  satisfying conditions of Proposition 3.1 is called unipotent.*

In this section we find a number of remarkable properties of unipotent traces which give a relatively simple procedure for the computation of their values on arbitrary elements of  $\mathcal{A}(\mathbf{GLB})$ . Let us sketch all these properties together first. Since  $\mathcal{A}(\mathbf{GLB})_n$  is isomorphic to the group algebra of  $\mathbf{GL}(n, q)$ , the traces can be viewed as functions on matrices from  $\mathbf{GL}(n, q)$  for various  $n$ . Such function is central, i.e. its values depend on the matrix through its Jordan normal form. One property of these functions is their multiplicativity which expresses the value on arbitrary Jordan normal forms as product of values on single block Jordan forms. Another property is a simple relation between values on the Jordan blocks with eigenvalue 1 and on Jordan blocks with arbitrary other eigenvalues. Final component is an expression for the values on the Jordan blocks with eigenvalue 1 in terms of specializations of *modified Hall-Littlewood polynomials*.

As for the restriction of unipotent trace on Hecke algebra  $\mathcal{H}_q(\infty) \subset \mathcal{A}(\mathbf{GLB})$ , in this section we identify them with extreme traces of  $\mathcal{H}_q(\infty)$  found in [VK89], see also [Me, Section 7], which also gives a formula for their values.

### 3.1. Values of unipotent traces: formulations.

**Theorem 3.3** (Multiplicativity theorem for unipotent traces). *Let  $\chi^\omega$  be an extreme unipotent trace. For  $g \in \mathbf{GL}(n, q)$  let  $\chi(g)$  denote the value of the restriction of  $\chi^\omega$  to  $\mathcal{A}(\mathbf{GLB})_n \simeq \mathbb{C}(\mathbf{GL}(n, q))$  on the element  $e_g \in \mathbb{C}(\mathbf{GL}(n, q))$ . Suppose that  $a \in \mathbf{GL}(n, q)$  and  $b \in \mathbf{GL}(m, q)$  are two matrices with coprime characteristic polynomials, then*

$$\chi^\omega(a)\chi^\omega(b) = \chi^\omega(a \odot b),$$

where  $a \odot b \in \mathbf{GL}(n + m, q)$  is the block-diagonal matrix with blocks  $a$  and  $b$ .

**Remark.** A very similar multiplicativity property holds for the extreme characters of the infinite symmetric group  $S(\infty)$  (see [Th64]) and infinite-dimensional unitary group  $U(\infty)$  (see [Vo]). This seems to be a general infinite-dimensional phenomenon.

The values of the unipotent traces on various conjugacy classes can be computed in terms of specializations of symmetric functions.

Let  $P_\mu(x_1, x_2, \dots; q^{-1})$  and  $Q_\mu(x_1, x_2, \dots; q^{-1})$  denote the Hall-Littlewood  $P$  and  $Q$  polynomials with parameter  $q^{-1}$  in variables  $x_1, \dots$  labeled by a Young diagram  $\mu$ , see [M, Chapter III]. Let  $\mathbb{M}_q$  denote the endomorphism of the algebra of symmetric functions

$$\mathbb{M}_q : \Lambda \rightarrow \Lambda$$

given on the Newton power sums  $p_k$  by the formula

$$p_k \rightarrow \frac{1}{1 - q^{-k}} p_k, \quad k \geq 0.$$

Denote

$$\tilde{P}_\mu = \mathbb{M}_q P_\mu, \quad \tilde{Q}_\mu = \mathbb{M}_q Q_\mu.$$

The symmetric functions  $\tilde{P}_\mu$  and  $\tilde{Q}_\mu$  are known as *modified Hall-Littlewood polynomials*.

**Theorem 3.4.** *Let  $\chi^\omega$  be an extreme unipotent trace. For  $g \in \mathbb{GL}(n, q)$  let  $\chi(g)$  denote the value of the restriction of  $\chi^\omega$  to  $\mathcal{A}(\mathbf{GLB})_n \simeq \mathbb{C}(\mathbb{GL}(n, q))$  on the element  $e_g \in \mathbb{C}(\mathbb{GL}(n, q))$ . Suppose that the characteristic polynomial of  $g$  is  $(x - 1)^n$  and the conjugacy class (i.e. Jordan Normal form of  $g$ ) is encoded by the Young diagram  $\lambda$  with  $n$  boxes. Then*

$$\chi(g) = q^{n(\lambda)} \mathcal{S}p_{\alpha, \beta, 1} [\tilde{Q}_\lambda],$$

where

$$n(\lambda) = \sum_i (i - 1) \lambda_i = \sum_i \binom{\lambda'_i}{2}.$$

Now let  $Pl_n$  be the endomorphism of the algebra  $\Lambda$  defined through

$$Pl_n : \Lambda \rightarrow \Lambda, \quad Pl_n(p_i) = p_{ni}$$

This is a particular case of *plethysm* morphism, see [M, Section I.8]. Note that  $Pl_n$  maps  $m_\lambda$  to  $m_{n\lambda}$ .

Now set

$$\mathcal{S}p_{\alpha, \beta, 1}^n := \mathcal{S}p_{\alpha, \beta, 1} \circ Pl_n.$$

Observe that

$$\mathcal{S}p_{\alpha, \beta, 1}^n [f] = \mathcal{S}p_{\alpha^n, -(-\beta)^n, 1} [f],$$

where  $\alpha^n = ((\alpha_1)^n, (\alpha_2)^n, \dots)$ ,  $-(-\beta)^n = (-(-\beta_1)^n, -(-\beta_2)^n, \dots)$ .

**Theorem 3.5.** *In the settings of Theorem 3.4 suppose that  $n = km$ , the characteristic polynomial of  $g$  is  $u^m$ , where  $u$  is an irreducible (over  $\mathbb{F}_q$ ) polynomial of degree  $k$  and the conjugacy class (i.e. Jordan Normal form of  $g$ ) is given by the Young diagram  $\lambda$  with  $m$  boxes. Then*

$$\chi(g) = q^{kn(\lambda)} \mathcal{S}p_{\alpha, \beta, 1}^k [\tilde{Q}_\lambda(\cdot; q^{-k})]$$

Clearly, combining Theorem 3.5 and Theorem 3.3 we get the formula for values of the unipotent traces on arbitrary conjugacy classes.

**3.2. Values of unipotent traces: proofs.** To give a proof of Theorem 3.3 we need some facts about the Hopf algebra related to the representations of  $\mathbb{GL}(n, q)$ . We follow [Zel] and [M] here.

Let  $D_n$  denote the space of central (i.e. class) complex functions on  $\mathbb{GL}(n, q)$ . This is a finite dimensional vector space with basis of the characteristic functions of the conjugacy classes of  $\mathbb{GL}(n, q)$ . The latter are parameterized by the elements of  $\mathcal{CY}_n$ .

Denote

$$D = \oplus_{n \geq 0} D_n.$$

The vector space  $D$  is a Hopf algebra with multiplication and comultiplication given by the operations of parabolic induction and parabolic restriction, respectively, see e.g. [Zel, Section 8, Chapter III] or [M, Section IV].

For a family  $f \in \mathcal{CY}_n$  let  $Cl_f \in D_n$  denote the indicator function of the conjugacy class in  $\mathbb{GL}(n, q)$  corresponding to  $f$ . The definition of the multiplication in  $D$  implies that for disjoint  $f_1, f_2$  we have

$$(3.1) \quad Cl_{f_1} \cdot Cl_{f_2} = Cl_{f_1+f_2}$$

Let  $R_n \subset D_n$  denote the  $\mathbb{Z}$ -module spanned by the characters of irreducible representations of  $\mathbb{GL}(n, q)$  and  $R = \bigoplus_{n \geq 0} R_n$ . Then  $D = R \otimes_{\mathbb{Z}} \mathbb{C}$ .  $R$  is a Hall subalgebra of  $D$ , moreover,  $R$  is a *PSH-algebra* in the terminology of [Zel].

**Proposition 3.6.** *We have*

$$(3.2) \quad R = \bigotimes_{c \in \mathcal{C}} R^c, \quad D = \bigotimes_{c \in \mathcal{C}} D^c,$$

where  $\bigotimes$  means the tensor product of Hopf algebras,  $R^c$  is  $\mathbb{Z}$ -module spanned by the characters of irreducible representations  $\chi^f$  of  $\mathbb{GL}(n, q)$  such that  $f(u) = \emptyset$  unless  $u = c$ .  $R^c$  is a Hopf subalgebra of  $R$ , elements  $c \in \mathcal{C}$  enumerate the so-called cuspidal irreducible representations of  $\mathbb{GL}(n, q)$ , which are in bijections with elements of  $\mathcal{C}$  and  $D^c = R^c \otimes_{\mathbb{Z}} \mathbb{C}$ . Each  $R^c$  is isomorphic to  $\Lambda$ , under this identification  $\chi^f \in R^c$  corresponds to  $s_{f(c)}$ .

*Proof.* See [Zel, 9.3]. □

Next we describe how the traces of  $\mathcal{A}(\mathbf{GLB})$  are related to algebra  $D$ . Let  $p^{\mathbf{GLB}}$  denote a distinguished degree one element of  $D$  which is the sum of characters of all  $q-1$  irreducible representation of  $\mathbb{GL}(1, q)$ .

Let  $\Xi^{\mathbf{GLB}}$  denote the convex cone of linear functionals on  $D$  satisfying:

$$\xi : D \rightarrow \mathbb{C}$$

- (1)  $\xi[u \cdot p^{\mathbf{GLB}}] = \xi[u]$ , for every  $u \in D$ ,
- (2)  $\xi[\chi] \geq 0$  for every  $n$  and every character  $\chi \in D_n$  of an irreducible representation of  $\mathbb{GL}(n, q)$ .

Let  $\phi \rightarrow \xi^\phi$  denote the map from the set of traces of  $\mathcal{A}(\mathbf{GLB})$  to  $\Xi^{\mathbf{GLB}}$  given by:

$$\xi^\phi[\chi^f] = c(f),$$

where  $\chi^f \in D_n$  is a character of irreducible representation of  $\mathbb{GL}(n, q)$  indexed by  $f \in \mathcal{CY}_n$  and  $c(f)$  is the coefficient in the decomposition

$$\phi|_{\mathcal{A}(\mathbf{GLB})_n}(\cdot) = \sum_{f \in \mathcal{CY}_n} c(f) \chi^f(\cdot).$$

Comparing the definitions of the set  $\Xi^{\mathbf{GLB}}$  and Proposition 3.6 with Theorem 2.13 we conclude that the map  $\phi \rightarrow \xi^\phi$  gives a bijection between the set of traces of  $\mathcal{A}(\mathbf{GLB})$  and  $\Xi^{\mathbf{GLB}}$ .

Moreover, note that for  $g \in \mathbb{GL}(n, q)$  belonging to a conjugacy class  $f$  we have (as follows from the definitions)

$$(3.3) \quad \phi(e_g) = \xi^\phi[Cl_f]$$

As and above we use the identification  $\mathcal{A}(\mathbf{GLB})_n \simeq \mathbb{C}(\mathbb{GL}(n, q))$  here.

Now we can prove Theorem 3.3.

*Proof of Theorem 3.3.* Let  $\tilde{\chi}$  (which is  $\chi^\omega$  for some  $\omega \in \Omega(\mathbf{GLB})$ ) be a unipotent extreme trace of  $A(\mathbf{GLB})$ . For  $b \in R^{“x-1”}$  the values of  $\xi^{\tilde{\chi}}[b]$  were computed in Theorem 2.26. In particular, since specializations  $\mathcal{S}p_{\alpha,\beta,\gamma}$  are algebra homomorphisms by their definition and  $R^{“x-1”} \simeq \Lambda$ , we have for  $b_1, b_2 \in R^{“x-1”}$

$$(3.4) \quad \xi^{\tilde{\chi}}[b_1 b_2] = \xi^{\tilde{\chi}}[b_1] \xi^{\tilde{\chi}}[b_2].$$

We claim that (3.2) actually holds for general  $b_1, b_2 \in D$ . To prove this claim note that both sides of (3.4) are bilinear. Therefore, it enough to check this property for  $b_1, b_2$  belonging to some linear basis of  $D$ . Let us choose the basis  $b^f$  enumerated by  $f \in \mathcal{CY}$  and given by

$$b^f := \chi^f = \prod_{c \in \mathcal{C}} \chi^{f_c},$$

where  $\chi^{f_c} \in R^c$  is the character of the irreducible representation of  $\mathbb{GL}(|f(c)|, q)$  corresponding to the family  $f_c$  defined as a unique family such that  $f_c(c) = f(c)$  and  $f_c(u) = \emptyset$  for  $u \neq c$ ;  $\chi^f$  as and above is the corresponding character of the irreducible representation of  $\mathbb{GL}(|f|, q)$ . It remains to observe that by the definition of the unipotent character

$$\xi^{\tilde{\chi}}[b^f] = 0$$

unless  $f(u) = 0$  for any  $u \neq “x-1”$ .

Now suppose that  $a \in \mathbb{GL}(n, q)$  and  $b \in \mathbb{GL}(m, q)$  are two matrices with coprime characteristic polynomials belonging to conjugacy classes parameterized by  $f \in \mathcal{CY}_n$  and  $h \in \mathcal{CY}_m$ , respectively. Then the families  $f$  and  $h$  are disjoint, furthermore,  $a \circ b$  belongs to the conjugacy class parameterized by  $f + h$ . Therefore, using (3.4), (3.1) and (3.3) we obtain

$$\chi^\omega(a \circ b) = \xi^{\chi^\omega}[Cl_{f+h}] = \xi^{\chi^\omega}[Cl_f \cdot Cl_h] = \xi^{\chi^\omega}[Cl_f] \xi^{\chi^\omega}[Cl_h] = \chi^\omega(a) \chi^\omega(b)$$

□

*Proof of Theorem 3.4.* If  $\chi^\mu$  is the character of the unipotent representation of  $\mathbb{GL}(n, q)$  indexed by  $\mu$ , then as follows e.g. from the results of [M, Chapter IV]

$$\chi^\mu(g) = q^{n(\lambda)} K_{\mu, \lambda}^{q^{-1}},$$

where  $K_{\mu, \lambda}^{q^{-1}}$  is the  $q^{-1}$  Kostka number defined as the coefficient in the decomposition of Schur polynomials into the sum of Hall-Littlewood polynomials

$$(3.5) \quad s_\mu(x_1, x_2, \dots) = \sum_{\lambda} K_{\mu, \lambda}^{q^{-1}} P_\lambda(x_1, x_2, \dots; q^{-1}).$$

Therefore,

$$(3.6) \quad \chi^\omega(g) = q^{n(\lambda)} \sum_{\mu} K_{\mu, \lambda}^{q^{-1}} \mathcal{S}p_{\alpha, \beta, 1}[s_\lambda].$$

It remains to prove that

$$(3.7) \quad \hat{Q}_\lambda = \sum_{\mu} K_{\mu, \lambda}^{q^{-1}} s_\mu.$$

Indeed, the Cauchy identity for Hall-Littlewood polynomials (see [M, Section III.4]) yields

$$(3.8) \quad \sum_{\nu \in \mathbb{Y}} P_\nu(x_1, x_2, \dots; q^{-1}) Q_\nu(y_1, y_2, \dots; q^{-1}) u = \prod_{i, j} (1 - q^{-1} x_i y_j) (1 - x_i y_j)$$

Applying the map  $M_q$  with respect to the variables  $y_1, y_2, \dots$  in the identity (3.8) we arrive at

$$(3.9) \quad \sum_{\nu \in \mathbb{Y}} P_\nu(x_1, x_2, \dots; q^{-1}) \tilde{Q}_\nu(y_1, y_2, \dots; q^{-1}) u = \prod_{i,j} (1 - x_i y_j)^{-1}$$

In the same time the Cauchy Identity for Schur Polynomials (see [M, Section I.4]) yields

$$(3.10) \quad \sum_{\nu \in \mathbb{Y}} s_\nu(x_1, x_2, \dots; q^{-1}) s_\nu(y_1, y_2, \dots; q^{-1}) u = \prod_{i,j} (1 - x_i y_j)^{-1}$$

Combining (3.9), (3.10) and (3.5) we arrive at (3.7).  $\square$

We need some preparations to prove Theorem 3.5. In order to connect the values of unipotent traces on unipotent and on more general conjugacy classes it is convenient to work not with irreducible representations of  $\mathbb{GL}(n, q)$ , but with representations induced from the parabolic subgroups.

More precisely, let  $\mu$  be a Young diagram with  $n$  boxes and let  $fl_\mu$  denote the set of all flags of subspaces

$$V_1 \subset V_2 \subset \dots \subset V_r$$

of  $n$ -dimensional vector space  $\mathbb{F}_q^n$ , such that  $\dim V_k = \mu_1 + \dots + \mu_k$ . Here  $r$  is the number of nonempty rows in  $\mu$ .  $\mathbb{GL}(n, q)$  naturally acts in  $fl_\mu$ . Let  $\psi_\mu^q$  denote the character of the corresponding representation of  $\mathbb{GL}(n, q)$  in  $\mathbb{C}(fl_\mu)$ . Clearly,  $\psi_\mu(g)$  is equal to the number of flags in  $fl_\mu$  which  $g$  fixes.

First, we claim that if the conjugacy class of  $g$  is given by a family  $f \in \mathbb{CY}_n$  such that  $f(p) = \emptyset$  for all but one linear polynomial  $p$ , then the value of  $\psi_\mu(g)$  does not depend on this  $p$  (but, of course, depends on the Young diagram  $f(p)$ ). Indeed, if  $g_1$  and  $g_2$  are two such matrices, then one can be obtained from another by conjugation and addition of a scalar matrix. Conjugation does not change the character. The addition of a scalar matrix does not change the set of invariant subspaces of an operator, thus, also does not change the character.

Next suppose that  $n = mk$  and let  $p(x)$  be an irreducible polynomial of degree  $k$ . Suppose that the conjugacy class of  $g \in \mathbb{GL}(n, q)$  is given by a family  $f$ ,  $f(\cdot)$  is equal to empty set everywhere except at  $p$  and  $f(p) = \lambda$ . This implies that  $|\lambda| = m$ . Suppose also that  $g'_y \in \mathbb{GL}(m, q^k)$  (note that the number of elements in the field changed!) is in a conjugacy class  $f'$  such that  $f(p') = \lambda$  for linear polynomial  $p'(x) = x - y$  (here  $y \in \mathbb{F}_{q^k}^*$ ). Our next aim is to prove that

$$(3.11) \quad \psi_\mu^q(g) = \begin{cases} \psi_\nu^{q^k}(g'_y), & \text{if } \mu_i = k\nu_i \text{ for all } i, \\ 0, & \text{otherwise.} \end{cases}$$

Note that (as we have shown above) the right side of (3.11), actually, does not depend on  $y$ .

Conjugating the matrix, if necessary, we can assume that  $g$  is made out of  $k \times k$  blocks. On the main diagonal all the blocks are  $Mat(p)$ , where

$$Mat(p) = \begin{pmatrix} 0 & 0 & 0 & \dots & -p_0 \\ 1 & 0 & 0 & \dots & -p_1 \\ & & \dots & & \\ 0 & \dots & 1 & 0 & -p_{k-2} \\ 0 & \dots & 0 & 1 & -p_{k-1} \end{pmatrix}$$

is the companion matrix of the polynomial

$$p(x) = p_0 + p_1x + \cdots + p_{k-1}x^{k-1} + x^k.$$

Below the diagonal all blocks are zero. Directly above the main diagonal blocks are either  $k \times k$  identity matrices or zeros. Identity matrices form (diagonal) groups of lengths (from top to bottom)  $\lambda_1, \lambda_2, \dots$

The matrix  $g'$  can be assumed to have a similar structure (but without any blocks), e.g.

$$g' = \begin{pmatrix} y & 1 & 0 & & & \\ 0 & y & 1 & 0 & & \\ 0 & 0 & y & 0 & 0 & \\ 0 & 0 & 0 & y & 1 & 0 \\ & & \cdots & & & \\ 0 & \dots & & & 0 & y \end{pmatrix}$$

More formally,  $g'$  has  $y$ s on the main diagonal, zeros everywhere below the main diagonal and above the second (i.e. the one on top of the main diagonal) and 1s in the second diagonal forming groups divided by zeros. In the above example the length of the first group, which is  $\lambda_1$ , equals 2.

Now let us view the  $n$ -dimensional space  $\mathbb{F}_q^n$  as  $(\mathbb{F}_q^k)^m$ . Let us identify  $\mathbb{F}_q^k$  with  $\mathbb{F}_{q^k}$ . The main step in proving (3.11) is the following lemma.

**Lemma 3.7.** *There exists  $y \in \mathbb{F}_{q^k}$  such that if  $V$  is a  $\mathbb{F}_q$ -linear subspace of  $(\mathbb{F}_q^k)^m$  invariant under  $g$ , then  $V$  is a  $\mathbb{F}_{q^k}$ -linear subspace invariant under  $g'_y$  and vice versa.*

*Proof.* Let  $h$  be  $n \times n$  matrix made out of  $m$   $k \times k$  blocks on the diagonal, each block is  $Mat(p)$ . If we set  $Q = q^t$  with large enough  $t$ , then  $(g - h)^Q = 0$ . Note that the matrices  $g$  and  $h$  commute, therefore  $(g - h)^Q = g^Q + (-h)^Q$ . We conclude that  $V$  is invariant under  $h^Q$ .

Now let us consider the field  $\mathbb{F}_q[x]/p(x) \simeq \mathbb{F}_{q^k}$ . Note that  $Mat(p)$  is the matrix of the operator of multiplication by  $x$  in the basis  $1, x, \dots, x^{k-1}$ . Since  $x$  now can be viewed as an element of the field with  $q^k$  elements, we conclude that  $Mat(p)^{q^k} = Mat(p)$ . Therefore, it is possible to choose large  $t$  so that  $Mat(p)^Q = Mat(p)$ . Thus,  $h^Q = h$  and  $V$  is invariant under  $h$ . Any element of  $\mathbb{F}_q[x]/p \simeq \mathbb{F}_{q^k}$  is a polynomial in  $x$ , therefore,  $V$  is invariant under the multiplicative group of  $\mathbb{F}_{q^k}$ . In other words,  $V$  is a  $\mathbb{F}_{q^k}$ -linear subspace. Now identifying  $x \in \mathbb{F}_q[x]/p(x)$  with  $y \in \mathbb{F}_{q^k}$  we see that  $V$  is invariant under the  $g'_y$ .

In the reverse direction the statement is immediate.  $\square$

Now (3.11) becomes straightforward. Indeed, the left side of (3.11) equals the number of  $g$ -invariant flags. If  $k$  does not divide  $\mu_i$  for some  $i$ , then Lemma 3.7 implies that there are simply no invariant flags. And if  $\mu_i = k\nu_i$  for all  $k$ , then the flags from  $fl_\mu$  are identified with flags from  $fl_\nu$  over bigger field  $\mathbb{F}_{q^k}$  and we arrive at the right side of (3.11).

*Proof of Theorem 3.5.* Now we can deduce the formula for the values of the unipotent traces. Decompose  $\chi^\omega$  into the sum of the characters  $\psi_\mu^q$ :

$$\chi^\omega = \sum_{\mu \in \mathbb{Y}_n} c_\mu \psi_\mu^q$$



In order to calculate the coefficients  $c_\mu$  we recall the decomposition of the characters of the irreducible unipotent representations of  $\mathbb{GL}(n, q)$  into the sum of  $\psi_\mu^q$ . We have

$$(3.12) \quad \chi^\lambda = \sum_{\mu} K_{\lambda, \mu} \psi_\mu^q,$$

where the coefficients  $K_{\lambda, \mu}$  are Kostka numbers and do not depend on  $q$  (see [St] and also [M, Section I.6] and references therein). They can be defined via the relations in the algebra of symmetric functions  $\Lambda$ :

$$m_\mu = \sum_{\lambda} K_{\lambda, \mu} s_\lambda$$

where  $m_\mu$  is the monomial symmetric function indexed by  $\mu$ . We have

$$\begin{aligned} \chi^\omega = \sum_{\lambda \in \mathbb{Y}_n} \chi^\lambda \mathcal{S}p_{\alpha, \beta, 1}[s_\lambda] &= \sum_{\mu \in \mathbb{Y}_n} \psi_\mu^q \left( \sum_{\lambda \in \mathbb{Y}_n} K_{\lambda, \mu} \mathcal{S}p_{\alpha, \beta, 1}[s_\lambda] \right) \\ &= \sum_{\mu \in \mathbb{Y}_n} \mathcal{S}p_{\alpha, \beta, 1}[m_\mu] \psi_\mu^q \end{aligned}$$

Evaluating in  $g$  and using (3.11) we get

$$\chi^\omega(g) = \sum_{\nu \in \mathbb{Y}_m} \mathcal{S}p_{\alpha, \beta, 1}[m_{k\nu}] \psi_\nu^{q^k}(g') = \sum_{\nu \in \mathbb{Y}_m} \mathcal{S}p_{\alpha, \beta, 1}^k[m_\nu] \psi_\nu^{q^k}(g')$$

Converting back into the sum of irreducible unipotent characters we get

$$\chi^\omega(g) = \sum_{\lambda \in \mathbb{Y}_m} \mathcal{S}p_{\alpha, \beta, 1}^k[s_\lambda] \chi^\lambda(g').$$

Now the application of Theorem 3.4 (with  $q$  replaced by  $q^k$ ) completes the proof.  $\square$

**3.3. Restriction of unipotent traces to Iwahori–Hecke algebra.** In this section we explain what happens when one restricts unipotent trace of  $\mathcal{A}(\mathbf{GLB})$  on  $\mathcal{H}_q(\infty)$ .

First, we recall a classical theorem relating representations of  $\mathcal{H}_q(n)$  and  $\mathbb{GL}(n, q)$ .

**Proposition 3.8.** *Both irreducible representations of  $\mathcal{H}_q(n)$  and unipotent irreducible representations of  $\mathbb{GL}(n, q)$  are parameterized by the set  $\mathbb{Y}_n$  of Young diagrams with  $n$  boxes. The representation of  $\mathcal{H}_q(n)$  indexed by  $\lambda$  coincides with the restriction of the representation of  $\mathbb{GL}(n, q)$  indexed by  $\lambda$  on the set of  $\mathbf{B}_n$ -invariant vectors and on the subalgebra  $\mathcal{H}_q(n) \subset \mathbb{C}(\mathbb{GL}(n, q))$ . In particular the restriction of the conventional character (matrix trace) of the irreducible unipotent representation of  $\mathbb{GL}(n, q)$  on  $\mathcal{H}_q(n)$  is the character of the corresponding irreducible representation of  $\mathcal{H}_q(n)$ .*

*Proof.* See e.g. [CF].  $\square$

The traces of infinite Hecke algebra  $\mathcal{H}_q(\infty)$  were first classified in [VK89], recently they were also studied in [Me]. From these articles the following result is known.

**Proposition 3.9.** *Extreme traces of  $\mathcal{H}_q(\infty)$  normalized by the condition  $\chi(s_e) = 1$  are enumerated by sequences  $\alpha = \{\alpha_i\}$ ,  $\beta = \{\beta_i\}$  satisfying*

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq 0, \quad \beta_1 \geq \beta_2 \geq \cdots \geq 0, \quad \sum_i (\alpha_i + \beta_i) \leq 1.$$

*The decomposition of the restriction of trace  $\chi^{\alpha, \beta}$  on  $\mathcal{H}_q(n)$  into traces  $\chi^\lambda$  of irreducible representations of  $\mathcal{H}_q(n)$  is given by the following formula:*

$$\chi^{\alpha, \beta}|_{\mathcal{H}_q(n)} = \sum_{\lambda \in \mathbb{Y}_n} \mathcal{S}p_{\alpha, \beta, 1}[s_\lambda] \chi^\lambda$$

Combining Propositions 3.8 and 3.9 with Theorem 2.26 we arrive at the following statement which should be viewed as an infinite-dimensional analogue of Proposition 3.8.

**Theorem 3.10** (Restriction theorem for unipotent traces). *The restriction of the extreme unipotent trace of  $\mathcal{A}(\mathbf{GLB})$  indexed by pair of sequences  $\alpha, \beta$  on the infinite-dimensional Hecke algebra  $\mathcal{H}_q(\infty) \subset \mathcal{A}(\mathbf{GLB})$  is the extreme trace of  $\mathcal{H}_q(\infty)$  indexed by the same parameters.*

#### 4. IDENTIFICATION OF UNIPOTENT TRACES WITH PROBABILITY MEASURES

For any  $g \in \mathbb{GL}(n, q)$  let  $\text{Cyl}_g^{\mathbf{GLB}}$  denote the set of all  $h \in \mathbf{GLB}$  such that  $I_g^{\mathbf{GLB}}(h) = 1$ . We call  $\text{Cyl}_g^{\mathbf{GLB}}$  the cylindrical set of  $g$ . Clearly, sets  $\text{Cyl}_g^{\mathbf{GLB}}$  span the  $\sigma$ -algebra of Borel sets on  $\mathbf{GLB}$ .

**Theorem 4.1.** *Let  $\chi^\omega$  be a unipotent trace of  $\mathcal{A}(\mathbf{GLB})$ . There exist a unique probability measure  $\varrho_\omega^{\mathbf{GLB}}$  on  $\mathbf{B} \subset \mathbf{GLB}$ , such that for any upper-triangular matrix  $g \in \mathbb{GL}(n, q)$  we have:*

$$(4.1) \quad \chi^\omega(I_g^{\mathbf{GLB}}) = \varrho_\omega^{\mathbf{GLB}}(\text{Cyl}_g^{\mathbf{GLB}})$$

*Proof.* The  $q$ -Kostka numbers  $K_{\mu, \lambda}^{q^{-1}}$  are polynomials in  $q^{-1}$  with non-negative coefficients (see e.g. [M, Section III.6]). Therefore,  $K_{\mu, \lambda}^{q^{-1}} \geq 0$  for all  $\mu$  and  $\lambda$ . Furthermore, also  $\mathcal{S}p_{\alpha, \beta, 1}[s_\lambda] \geq 0$  (see [VK90], [K03]). Therefore, formula (3.6) implies that the values in the left side of (4.1) are non-negative. Consequently, we can define the measure  $\varrho_\omega^{\mathbf{GLB}}$  through the equation (4.1). Definitions of the functions  $I_g^{\mathbf{GLB}}$  and traces readily imply that we get a well-defined probability measure.

The uniqueness follows from Theorems 3.5 and 3.3 which prove that the trace  $\chi^\omega$  is uniquely defined by its values on unipotent classes.  $\square$

**Remark.** Analyzing the statement of Theorem 3.5 one sees that, for general  $g$ , the values of  $\chi^\omega(I_g^{\mathbf{GLB}})$  might be negative. Thus, if we try to extend the measure  $\varrho^\omega$  to the whole group  $\mathbf{GLB}$ , then it will no longer be a positive measure.

The aim of this section is to analyze the properties of measures  $\varrho_\omega^{\mathbf{GLB}}$ . At this point it is convenient to switch from  $\mathbf{GLB}$  to  $\mathbf{GLU}$ . While all the proofs remain almost the same, but the statements for  $\mathbf{GLU}$  look simpler and more clear. The reason is that unipotent upper triangular matrices have a unique eigenvalue 1, while general upper-triangular matrices might have up to  $q - 1$  different eigenvalues and we would have to analyze the part of measure corresponding to each of them separately.

The whole theory for  $\mathbf{GLU}$  is very much parallel to  $\mathbf{GLB}$ . We give an overview here, the details can be found in the Appendix. In the same way as for  $\mathbf{GLB}$  we introduce the algebra  $\mathcal{A}(\mathbf{GLU})$  of continuous functions on  $\mathbf{GLU}$  with compact support taking only finitely many values.  $\mathcal{A}(\mathbf{GLU})$  is yet again an inductive limit of algebras  $\mathcal{A}(\mathbf{GLU})_n$  isomorphic to  $\mathbb{C}(\mathbb{GL}(n, q))$ , however, the embeddings become different. While the classification of traces of  $\mathcal{A}(\mathbf{GLU})$  is a bit different than that of  $\mathcal{A}(\mathbf{GLB})$  there is still a class of unipotent traces parameterized by sequences  $\alpha, \beta$ . Moreover, under the identification  $\mathcal{A}(\mathbf{GLU})_n \simeq \mathbb{C}(\mathbb{GL}(n, q)) \simeq \mathcal{A}(\mathbf{GLB})_n$  the restriction of unipotent trace of  $\mathcal{A}(\mathbf{GLU})$  and  $\mathcal{A}(\mathbf{GLB})$  are the same functions. Because of that it makes no reason to distinguish between the unipotent traces of  $\mathcal{A}(\mathbf{GLU})$  and  $\mathcal{A}(\mathbf{GLB})$ .

For  $\mathbf{GLU}$  the group  $\mathbf{B}$  is replaced by the subgroup  $\mathbf{U}$  of unipotent upper-triangular matrices and Theorem 4.1 transforms into Theorem 4.2 (with notions of the indicator function  $I_g^{\mathbf{GLU}}$  and cylindrical set  $\text{Cyl}_g^{\mathbf{GLU}}$  analogous to those for  $\mathbf{GLB}$ ) the proof of which remains the same.

**Theorem 4.2.** *Let  $\chi^\omega$  be a unipotent trace of  $\mathcal{A}(\mathbf{GLU})$ . There exist a unique probability measure  $\varrho_\omega^{\mathbf{GLU}}$  on  $\mathbf{U}$ , such that for any upper-triangular matrix  $g \in \mathbb{GL}(n, q)$  we have:*

$$(4.2) \quad \chi^\omega(I_g^{\mathbf{GLU}}) = \varrho_\omega^{\mathbf{GLU}}(\text{Cyl}_g^{\mathbf{GLU}})$$

The measures corresponding to unipotent traces belong to a more general class of measures which we now describe.

**Definition 4.3.** *A probability measure  $\varrho$  on  $\mathbf{U}$  is called central if  $\varrho(\text{Cyl}_g^{\mathbf{GLU}})$  depends only on the conjugacy class, i.e. on the Jordan normal form of  $g$ . In other words,  $\varrho$  is invariant under conjugations by elements of  $\mathbb{GL}(\infty, q)$ .*

**Remark.** Conjugations, in general, do not preserve the set of upper-triangular matrices, so the invariance means that if both  $M \subset U$  and  $gMg^{-1} \subset U$  for some measurable set  $M$  and  $g \in \mathbb{GL}(\infty, q)$ , then the measures of  $M$  and  $gMg^{-1}$  are equal.

**Definition 4.4.** *A central measure  $\varrho$  is called ergodic if it is an extreme point of the convex set of all central probability measures.*

The following conjecture describes the set of all ergodic central measures on  $\mathbf{U}$ . For a matrix  $u \in \mathbf{U}$ , as above,  $u^{(n)}$  stays for its top-left  $n \times n$  corner. Since all the eigenvalues of  $u$  are 1s, the Jordan normal form of  $u^{(n)}$  can be parameterized by a Young diagram  $\lambda$ , let  $\lambda_i(u, n)$  and  $\lambda'_i(u, n)$  denote the row and column lengths of  $\lambda$ , respectively.

**Conjecture 4.5** (Classification and law of large numbers for ergodic central measures). *Let  $\vartheta$  be an ergodic central measure on  $\mathbf{U}$ . There exist two sequences  $r_i, c_i$  (row frequencies and column frequencies), such that for every  $i$ ,  $\vartheta$ -almost surely*

$$\lim_{n \rightarrow \infty} \frac{\lambda_i(u, n)}{n} = r_i, \quad \lim_{n \rightarrow \infty} \frac{\lambda'_i(u, n)}{n} = c_i.$$

*Moreover, for each pair of sequences  $r = (r_1 \geq r_2 \geq \dots \geq 0)$  and  $c = (c_1 \geq c_2 \geq \dots \geq 0)$  satisfying  $\sum_i (r_i + c_i) \leq 1$  there exists a unique ergodic central measure  $\vartheta^{r, c}$  with corresponding row and column frequencies.*

The  $\vartheta^{r,c}$ -probabilities of cylindrical sets can be expressed through the row frequencies and column frequencies via the formula

$$(4.3) \quad \vartheta(\text{Cyl}_g^{\text{GLU}}) = \frac{q^{-n(n-1)/2}}{(1-q^{-1})^n} q^{n(\lambda)} \mathcal{S}p_{r,c^{(q)},1}[Q_\lambda(\cdot; q^{-1})]$$

where  $g \in \text{GL}(n, q)$  is an unipotent upper-triangular matrix corresponding to the conjugacy class  $\lambda$ ,  $Q_\lambda$  is, as above, the Hall-Littlewood polynomial and for a sequence  $c = c_1, c_2, \dots$ , the sequence  $c^{(q)}$  is obtained by rearranging two-dimensional array of numbers  $(1 - q^{-1})c_i q^{1-j}$ ,  $i, j = 1, 2, \dots$  in decreasing order.

**Remark 1.** Conjecture 4.5 is a particular case of the conjecture on Macdonald polynomials stated in [K03, Section II.9] and which is now known as Kerov conjecture. A part of this conjecture was also briefly mentioned in Section 4 of [Fu].

**Remark 2.** If row frequencies  $r_i$  form a geometric series  $(1 - q^{-1}), (1 - q^{-1})q^{-1}, \dots$  and column frequencies  $c_i$  are zero, then (see [M, Exercise 1 in Section III.2])

$$\mathcal{S}p_{r,c^{(q)},1}[Q_\lambda(\cdot; q^{-1})] = (1 - q^{-1})^n q^{-n(\lambda)}$$

and  $\vartheta^{r,c}$  becomes the Haar (put if otherwise, uniform) measure on  $U$ . The row and column frequencies for the Haar measure on  $\mathbf{U}$  were first found by A. Borodin in [B1], [B2].

**Remark 3.** If  $r_i = 0$  and  $c = (1, 0, 0, \dots)$  then  $\vartheta^{r,c}$  is the delta-measure on the identity matrix.

**Remark 4.** If  $r = (1, 0, \dots)$  and  $c_i = 0$  then  $\vartheta^{r,c}$  is the uniform measure on matrices  $u \in \mathbf{U}$  such that  $u - Id$  has maximal possible rank. In other words, all matrix elements of  $u$  on the second diagonal are non-zero.

The existence of row and column frequencies for the ergodic central measure can, in principle, be deduced from the ergodicity. The hard (and still open) part of Conjecture 4.5 is the fact that the frequencies uniquely define the measure. Below we prove two partial results towards Conjecture 4.5, in particular, we show that the measure  $\vartheta^{r,c}$  with cylindrical probabilities given by (4.3) is, indeed, an ergodic central measure. But, first, let us explain the relation of measures  $\vartheta^{r,c}$  to the unipotent traces.

**Theorem 4.6.** *The measure  $\varrho_\omega^{\text{GLU}}$  is an ergodic central probability measure on  $\mathbf{U}$ . More precisely, if  $\omega = (\alpha, \beta)$ , then  $\varrho^\omega = \vartheta^{\alpha^{(q)}, \beta}$ , where for a sequence  $\alpha = \alpha_1, \alpha_2, \dots$ , the sequence  $\alpha^{(q)}$  is obtained by rearranging two-dimensional array of numbers  $(1 - q^{-1})\alpha_i q^{1-j}$ ,  $i, j = 1, 2, \dots$  in decreasing order.*

**Remark.** Note the *dual* role of row frequencies. On one hand, the parameters  $\alpha_i, \beta_i$  of unipotent characters are limit row frequencies of Young diagrams parameterizing irreducible unipotent representations of  $\text{GL}(n, q)$ , see [VK81], [KOO]. On the other hand frequencies show up in the limit behavior of Jordan Normal forms. The conceptual explanation of this double appearance of frequencies is unknown yet. Somehow similar phenomena is present in the asymptotic representation theory of symmetric groups with certain explanation given by the RSK algorithm, see [KV86].

*Proof of Theorem 4.6.* Theorem 3.5 implies that the cylindrical probabilities of measure  $\varrho^\omega$  are given by.

$$(4.4) \quad \varrho^\omega(\text{Cyl}_g^{\text{GLU}}) = q^{-n(n-1)/2} q^{n(\lambda)} \mathcal{S}p_{\alpha, \beta, 1} [\tilde{Q}_\lambda(\cdot; q^{-1})]$$

Note that

$$\mathcal{S}p_{\alpha, \beta, 1} [\tilde{Q}_\lambda(\cdot; q^{-1})] = (1 - q^{-1})^{-|\lambda|} \mathcal{S}p_{\alpha^{(q)}, \beta^{(q)}} [Q_\lambda(\cdot; q^{-1})].$$

Comparing (4.4) with (4.3) we conclude that  $\varrho^\omega = \vartheta^{\alpha^{(q)}, \beta}$ .  $\square$

Now let us prove two results related to Conjecture 4.5.

**Proposition 4.7.** *For any sequences  $r = \{r_i\}$ ,  $c = \{c_i\}$  satisfying  $\sum_i (r_i + c_i) \leq 1$  the measure  $\vartheta^{r, c}$  with cylindrical probabilities (4.3) is an ergodic central measure on  $\mathbf{U}$ .*

*Proof.* The key property which we use, is the positivity of the structural constants of the multiplication in the basis of Hall-Littlewood polynomials. In other words, in the identity

$$(4.5) \quad P_\lambda(\cdot; q^{-1}) P_\mu(\cdot; q^{-1}) = \sum_\nu c_{\lambda, \mu}^\nu P_\nu(\cdot; q^{-1})$$

When  $q > 1$  all the coefficients  $c_{\lambda, \mu}^\nu$  are non-negative. This fact follows from the known formulas for these coefficients (see e.g. [Ra, Theorem 4.9], [Sc, Theorem 1.3], [KM] and references therein). Since Hall-Littlewood  $P$ -polynomials and  $Q$ -polynomials differ by the multiplication by the constant, which is positive for  $q > 1$  (see [M, Section III.2]) we can replace  $P$  by  $Q$  in any part of (4.5) and the coefficients will be still positive. Moreover, (4.5) is equivalent to the equality for skew Hall-Littlewood polynomials (see [M, Section III.5])

$$(4.6) \quad Q_{\nu/\mu} = \sum_\lambda c_{\lambda, \mu}^\nu Q_\lambda$$

Again if we replace  $Q$  with  $P$  in either sides of (4.5) then the coefficients remain positive.

Let us prove that for any sequences  $r_i, c_i$ , the values

$$\mathcal{S}p_{r, c^{(q)}, 1} [Q_\lambda(\cdot; q^{-1})]$$

are nonnegative, which will guarantee the non-negativity of probabilities in (4.3).

First, let  $r = (1, 0, 0, \dots)$ ,  $c = (0, 0, \dots)$ . Then  $\mathcal{S}p_{r, c^{(q)}, 1} P_\lambda(\cdot; q^{-1}) = 0$  unless  $\lambda$  is a one-row diagram, i.e.  $\lambda_1 = n$ ,  $\lambda_2 = 0$ . In the latter case  $\mathcal{S}p_{r, c^{(q)}, 1} [P_\lambda(\cdot; q^{-1})] = 1$ . Thus,  $\mathcal{S}p_{r, c^{(q)}, 1} [Q_\lambda(\cdot; q^{-1})]$  is non-negative for all  $\lambda$ .

Second, let  $r = (0, 0, \dots)$ ,  $c = (1, 0, 0, \dots)$ . The Cauchy identity for Hall-Littlewood polynomials (see [M, Section III.4]) yields (here  $z$  and  $y$  stay for two sets of variables)

$$\sum_\lambda P_\lambda(z; q^{-1}) Q_\lambda(y; q^{-1}) = \exp \left( \sum_{m=1}^{\infty} \frac{1 - q^{-m}}{m} p_m(z) p_m(y) \right).$$

Applying  $\mathcal{S}p_{r,c(q),1}$  with respect to the variables  $y$  we get

$$\begin{aligned} \sum_{\lambda} P_{\lambda}(z; q^{-1}) \mathcal{S}p_{r,c(q),1}[Q_{\lambda}(y; q^{-1})] &= \exp \left( \sum_{m=1}^{\infty} \frac{(-1)^{m-1} (1 - q^{-1})^m}{m} p_m(z) \right) \\ &= \sum_{m \geq 0} (1 - q^{-1})^m e_m(z) \end{aligned}$$

Since  $P_{\lambda} = e_{|\lambda|}$  for one-column Young diagram  $\lambda = (1, \dots, 1)$ , we conclude that  $\mathcal{S}p_{r,c(q),1}[Q_{\lambda}(y; q^{-1})]$  is zero unless  $\lambda$  is a one-column diagram. In the latter case this number is positive.

Third, if  $r = c = (0, 0, \dots)$ , then  $\mathcal{S}p_{r,c(q),1}[Q_{\lambda}(y; q^{-1})]$  is precisely the coefficient of  $p_1^{|\lambda|}$  in the decomposition of  $Q_{\lambda}$  into the sum of products of power sums  $p_k$ . These coefficients are known to be non-negative, see [M, Exercise 4, Section III.7].

Next, suppose that we have two specializations  $\mathcal{S}p_1$  and  $\mathcal{S}p_2$  of  $\Lambda$  which map Hall-Littlewood polynomials to non-negative numbers. Take two nonnegative numbers  $a_1, a_2$  and consider a new specialization  $\mathcal{S}p$ , which we call *mixing* of  $\mathcal{S}p_1$  and  $\mathcal{S}p_2$ , given by

$$\mathcal{S}p[p_k] = (a_1)^k \mathcal{S}p_1[p_k] + (a_2)^k \mathcal{S}p_2[p_k].$$

We claim that the values of  $\mathcal{S}p$  on Hall-Littlewood polynomials are also nonnegative. Indeed, this follows from the identity (see [M, Section III.5])

$$\mathcal{S}p[Q_{\lambda}(\cdot; q^{-1})] = \sum_{\mu} (a_1)^{|\mu|} \mathcal{S}p_1[Q_{\mu}(\cdot; q^{-1})] (a_2)^{|\lambda| - |\mu|} \mathcal{S}p_2[Q_{\lambda/\mu}(\cdot; q^{-1})]$$

and the positivity of the coefficients in (4.6).

Now observe that starting with three simplest specializations which we described above, one can obtain any specialization  $\mathcal{S}p_{r,c(q),1}$  with finitely many non-zero  $r_i$ s and  $c_i$ s through mixing. Passing to the limit we conclude that  $\mathcal{S}p_{r,c(q),1}[Q_{\lambda}]$  is non-negative for all  $\lambda$  and all sequences  $r_i, c_i$  satisfying  $\sum_i (r_i + c_i) \leq 1$ .

Next, let us show that the central probability measures on  $U$  are in bijections with linear functionals

$$\phi : \Lambda \rightarrow \mathbb{C}$$

satisfying three coherency properties

- (1)  $\phi[Q_{\lambda}(\cdot; q^{-1})] \geq 0$  for every  $\lambda$
- (2)  $\phi[p_1 f] = \phi[f]$  for any  $f \in \Lambda$
- (3)  $\phi[1] = 1$ .

The correspondence is pretty much given by formula (4.3) and we keep the same notations, i.e. given a measure  $\vartheta$  the corresponding functional  $\phi_{\vartheta}$  is

$$(4.7) \quad \vartheta(\text{Cyl}_g^{\text{GLU}}) = \frac{q^{-n(n-1)/2}}{(1 - q^{-1})^n} q^{n(\lambda)} \phi_{\vartheta}[Q_{\lambda}(\cdot; q^{-1})].$$

The coherency properties 1. and 3. easily translate into the properties of a central probability measure. Let us deal with the coherency property 2.

By the very definition, the cylindrical probabilities of measure  $\vartheta$  should satisfy

$$\vartheta(\text{Cyl}_g^{\text{GLU}}) = \sum_{h \in \text{Ext}^{\text{GLU}}(g)} \vartheta(\text{Cyl}_h^{\text{GLU}}),$$



where

$$Ext^{\mathbf{GLU}}(g) = \left\{ [h_{ij}] \in \mathbb{GL}(n+1, q) \mid \right. \\ \left. h^{(n)} = g \text{ and } h_{n,1} = h_{n,2} = \cdots = h_{n,n-1} = 0, h_{n,n} = 1 \right\}$$

is an analogue of  $Ext^{\mathbf{GLB}}(g)$  of Section 2. Let us divide  $Ext^{\mathbf{GLU}}(g)$  into the groups having the same conjugacy class. We use the formula from [B2] which says that if conjugacy class of  $g$  is given by the Young diagram  $\lambda \in \mathbb{Y}_n$ , then the number  $N_{\lambda,\mu}$  of  $h \in Ext^{\mathbf{GLU}}(g)$  belonging to the conjugacy class given by the Young diagram  $\mu \in Y_{n+1}$  is

$$N_{\lambda,\mu} = \begin{cases} q^n q^{-\lambda'_j} (1 - q^{\lambda'_j - \lambda'_{j-1}}), & \text{if } \mu \setminus \lambda = \square_j, \\ 0, & \text{otherwise.} \end{cases}$$

Here  $\mu \setminus \lambda = \square_j$  means that the set-theoretical difference of the Young diagrams  $\mu$  and  $\lambda$  is a box in column  $j$  and we agree that  $\lambda_0 = +\infty$ , i.e.  $q^{\lambda'_1 - \lambda'_0} = 0$ .

Therefore, the functional  $\phi_\vartheta$  defined through (4.7) satisfies

$$(4.8) \quad \phi_\vartheta [Q_\lambda(\cdot; q^{-1})] = \sum_{\mu \in \mathbb{Y}_{n+1}} \phi_\vartheta [Q_\mu(\cdot; q^{-1})] N_{\lambda,\mu} q^{n(\mu) - n(\lambda)} \frac{q^{-n}}{1 - q^{-1}}$$

In the same time Pierry rules for the Hall-Littlewood polynomials (see [M, Section III.5]) yield

$$Q_\lambda(\cdot; q^{-1}) p_1 = \sum_{\mu \in Y_{n+1}: \mu \setminus \lambda = \square} (1 - q^{\lambda'_j - \lambda'_{j-1}}) Q_\mu(\cdot; q^{-1}),$$

where  $j$  is again the column of the box  $\mu \setminus \lambda$ . Therefore, (4.8) is equivalent to

$$\phi(\vartheta) [Q_\lambda(\cdot; q^{-1})] = \phi(\vartheta) [p_1 Q_\lambda(\cdot; q^{-1})].$$

Since the latter equality holds for every  $\lambda$  and Hall-Littlewood polynomials  $Q_\lambda(\cdot; q^{-1})$  form a linear basis of  $\Lambda$ , we arrive at the coherency property 2.

Now we are in position to use the so-called *Ring Theorem* (see [KV80], [VK90], [K03] and also [GO1, Section 8.7]). This theorem yields, that the extreme points of the convex set of  $Q_\lambda$ -positive functionals on  $\Lambda$  are those functional which are multiplicative (i.e. are algebra homomorphism). By the very definition the functionals  $Sp_{r,c(q),1}$  are multiplicative, they also satisfy the coherency properties. We conclude that these functionals are extreme and, thus, the corresponding measures  $\vartheta^{r,c}$  are indeed ergodic central measures on  $\mathbf{U}$ .  $\square$

**Proposition 4.8.** *If  $\vartheta$  is an ergodic central measure on  $\mathbf{U}$ , then there exist sequences  $\alpha, \beta$  satisfying (2.2) with  $\gamma = 1$  such that the measure  $\vartheta$  has the following cylindrical probabilities:*

$$\vartheta^{\alpha,\beta}(\text{Cyl}_g^{\mathbf{GLU}}) = \frac{q^{-n(n-1)/2}}{(1 - q^{-1})^n} q^{n(\lambda)} Sp_{\alpha,\beta,1}[Q_\lambda(\cdot; q^{-1})]$$

where  $g \in \mathbb{GL}(n, q)$  is an unipotent upper-triangular matrix corresponding to the conjugacy class  $\lambda$ .

**Remark.** Thus, to prove that the measures  $\vartheta^{r,c}$  exhaust the list of ergodic central measures it remains to show that if the sequence  $\beta$  is not a union of geometric series (with denominator  $q^{-1}$ ), then  $\mathcal{S}p_{\alpha,\beta,1} [Q_\lambda(\cdot; q^{-1})] < 0$  for some  $\lambda$ .

*Proof of Proposition 4.8.* As in the proof of Proposition 4.7 we identify ergodic central measures on  $\mathbf{U}$  with multiplicative functionals on  $\Lambda$  satisfying three coherency properties. Observe that the coefficients of the decomposition

$$s_\lambda(\cdot) = \sum_{\mu} c_{\lambda,\mu} Q_\lambda(\cdot; q^{-1})$$

are non-negative. Indeed, up to the simple constants they coincide with  $q$ -Kostka numbers (see [M, Section III.6]).

Therefore any multiplicative functional  $\phi$  satisfying three coherence properties also satisfy

$$\phi[s_\lambda] \geq 0.$$

But classification of the multiplicative functionals which are non-negative on Schur functions is well-known. It is equivalent to the Thoma theorem on the characters of infinite symmetric group  $S(\infty)$ , see [Th64], [VK81], [VK90], [K03]. The list of the functionals is given by  $\mathcal{S}p_{\alpha,\beta,1}$  with  $\alpha, \beta$  satisfying (2.2) with  $\gamma = 1$ .  $\square$

## 5. GRUPPOID CONSTRUCTION FOR THE REPRESENTATIONS OF $\mathbf{GLB}$

In this section we give an explicit construction for the representations of  $\mathbf{GLB}$  corresponding to a large class of the extreme unipotent traces of  $\mathcal{A}(\mathbf{GLB})$ .

**5.1. Generalities.** Generally speaking, we are going to construct irreducible *generalized spherical representations* of pair  $(\mathbf{GLB} \times \mathbf{GLB}, \mathbf{GLB})$ . Let us introduce some definitions first.

The well-known principle (see e.g. [D, Section 13]) identifies unitary representations of a locally-compact group  $G$  with  $*$ -representations of  $L_1(G)$  (with respect to Haar measure) and we will silently use this identification where it leads to no confusions.

**Definition 5.1.** A *generalized spherical representation* of  $(\mathbf{GLB} \times \mathbf{GLB}, \mathbf{GLB})$  is a triplet:

- (1) Unitary (continuous) representation  $\pi$  in a Hilbert space  $H$  with scalar product  $\langle \cdot, \cdot \rangle$ :

$$\pi : \mathbf{GLB} \times \mathbf{GLB} \rightarrow U(H),$$

- (2) Dense subspace  $H_1 \subset H$  equipped with a norm  $|\cdot|$ ,
- (3) Linear functional (distribution)  $v \in H_1'$

Satisfying the following conditions:

- (1) The inclusion  $i : (H_1, |\cdot|) \rightarrow (H, \langle \cdot, \cdot \rangle)$  is continuous,
- (2) For any  $(g, h) \in \mathbf{GLB} \times \mathbf{GLB}$ ,  $\pi(g, h)H_1 \subset H_1$
- (3) For any  $a \in \mathcal{A}(\mathbf{GLB})$  we have  $\pi(a, e)v \in H_1$  and  $\pi(e, a)v \in H_1$ . In other words, there exists  $w(a)$  such that  $\langle v, \pi(a, e)x \rangle = \langle w(a), x \rangle$  for every  $x \in H_1$ , and similarly for  $(e, a)$ .
- (4) The span of  $\{\pi(a, b)v \mid a, b \in \mathcal{A}(\mathbf{GLB})\}$  is dense in  $H$
- (5) For any  $g \in \mathbf{GLB}$ , we have  $\pi(g, g)v = v$ .
- (6)  $(\pi(I_{\mathbf{B}})v, v) = 1$ .

This definition is just a generalization of the well-known definition of a spherical representations of the Gelfand pair. The theory of spherical representations for the infinite symmetric group  $S(\infty)$  and infinite-dimensional unitary group  $U(\infty)$  was developed by G.Olshanski and his collaborators, see [O1], [O2], [O3]. The novelty in the present paper is the fact that the distinguished vector  $v$  no longer belongs to the Hilbert space, but becomes a distribution.

**Definition 5.2.** *A spherical function of a generalized spherical representation  $(\pi, H_1, v)$  is defined as*

$$\chi(a) = \langle \pi(a, b)v, v \rangle,$$

where either  $a$  or  $b$  belong to  $\mathcal{A}(\mathbf{GLB})$ .

Clearly, the restriction of the spherical function to its first coordinate (i.e. to pairs  $(a, e)$ ) gives a trace of  $\mathcal{A}(\mathbf{GLB})$ .

The converse is also true, i.e. given a trace of  $\mathcal{A}(\mathbf{GLB})$  we can, in principle, construct the corresponding representation (using a version of the so-called Gelfand-Naimark-Segal construction). However, the general construction is quite abstract and we seek for an explicit description of the representations corresponding to the unipotent traces  $\chi^\omega$ .

We could avoid the notion of a general spherical representation and use von Neuman factors instead, however, our approach seems to show more hidden structure. The following simple proposition explains how to pass to the factor-representations.

**Proposition 5.3.** *Let  $(\pi, H_1, v)$  be a generalized spherical representation such that the corresponding traces  $\chi$  of  $\mathcal{A}(\mathbf{GLB})$  is extreme. The restriction of  $\pi$  on the first coordinate is von Neumann semifinite (i.e. either type I or type II) factor representation of group  $\mathbf{GLB}$  in the cyclic span of  $v$ .*

*Proof.* Let  $\mathcal{V} \subset \mathcal{B}(H)$  denote the minimal von Neumann algebra containing all operators  $\pi(g, e)$ ,  $g \in \mathbf{GLB}$ . Let  $\chi'$  denote the (unique) extension of the trace  $\chi$  of  $\mathcal{A}(\mathbf{GLB})$  on  $\mathcal{V}$ . Clearly,  $\chi'$  is a semifinite trace of  $\mathcal{V}$ . Note that  $\chi'$  is extreme. Indeed, if  $\chi' = \chi'_1 + \chi'_2$ , then  $\chi$  has a similar decomposition and we get a contradiction with extremality of  $\chi$ . Extremality of  $\chi'$  implies that  $\mathcal{V}$  is a von Neumann factor.  $\square$

**Remark.** As we will see below the actual type of the factor representation can be different.

**5.2. Two simplest type I examples.** Recall that unipotent representations are parameterized by two sequences  $\alpha_1 \geq \alpha_2 \geq \dots \geq 0$  and  $\beta_1 \geq \beta_2 \geq \dots \geq 0$  such that  $\sum_i (\alpha_i + \beta_i) \leq 1$ . We start from considering some simplest cases.

First, suppose that  $\alpha_1 = 1$  with all other parameters being zeros. In this case the desired representation is just an identity representation. I.e.  $H$  is 1-dimensional vector space,  $H_1 = H$ ,  $\pi$  maps all elements of  $\mathbf{GLB} \times \mathbf{GLB}$  to identity operator and  $v$  is a unit vector in  $H$ .

Next, let  $\beta_1 = 1$ , and let all other parameters be zeros.

Let  $St_n$  be the Steinberg representation of  $\mathbb{GL}(n, q)$  (see [St], [Hu]). This representation can be realized as the left representation of  $\mathbb{GL}(n, q)$  in the right ideal of  $\mathbb{C}(\mathbb{GL}(n, q))$  spanned by the element

$$s = \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma e_\sigma \sum_{g \in \mathbf{B}_n} e_g$$

It is well known that a linear basis of  $H(St_n)$  can be chosen to be

$$\{e_g s \mid g \in \mathbf{U}_n\},$$

where  $\mathbf{U}_n$  is a subgroup of unipotent upper triangular matrices in  $\mathbb{GL}(n, q)$ . The dimension of  $H(St_n)$  is  $q^{n(n-1)/2}$ .

The representation  $St_{n-1}$  of  $\mathcal{A}(\mathbf{GLB})_{n-1}$  is naturally included into the representation  $St_n$  of  $\mathcal{A}(\mathbf{GLB})_n$  as the subspace of  $U_n^n$ -invariant vectors (see Theorem 2.11 for the definition of the group  $U_n^n$ ). Let  $St_\infty^0$  denote the inductive limit of the representations  $St_n$  with respect to the above embeddings. Note that each  $H(St_n)$  has a (unique up to a multiplication by a constant)  $\mathbb{GL}(n, q)$ -invariant scalar product and these scalar products can be chosen to agree with the above embeddings. Thus, the space  $H(St_\infty^0)$  is equipped with a scalar product. Let  $H(St_\infty)$  denote  $*$ -representation of  $\mathcal{A}$  in the completion of the space  $H(St_\infty^0)$ .

Now let  $\mathcal{H}$  denote the Hilbert space  $H(St_\infty)^* \otimes H(St_\infty)$  of Hilbert-Schmidt operators in  $H(St_\infty)$ . We have a natural  $*$ -representation  $\pi$  of  $\mathcal{A} \times \mathcal{A}$  in  $\mathcal{H}$ . It can be extended to a non-degenerate representation of  $L_1(\mathbf{GLB}) \otimes L_1(\mathbf{GLB})$  and, thus, to a unitary representation of  $\mathbf{GLB} \times \mathbf{GLB}$ , which we denote by the same letter  $\pi$ . Let  $\mathcal{H}_1 \subset \mathcal{H}$  be the subspace of trace-class operators and let the functional  $v \in \mathcal{H}'_1$  be trace. (If we identify  $\mathcal{H}'_1$  with the space  $\mathcal{B}(\mathcal{H})$  of bounded linear operators, then  $v$  corresponds to the identity operator.) Note that if  $a \in \mathcal{A}_n$ , then the image of the operator  $St_\infty(a)$  lies in  $H(St_n)$ . Therefore,  $St_\infty(a)$  has finite rank. It follows that  $\pi((a, b))v \in \mathcal{H}_1$  for any  $(a, b) \in \mathcal{A} \times \mathcal{A}$ . All other properties are trivial and we conclude that  $(\mathcal{H}, \mathcal{H}_1, v)$  is a generalized spherical representation. One immediately checks that the spherical function of this representation corresponds to the trace with  $\beta_1 = 1$  and all other parameters being zeros.

**5.3. Representations related to grassmanian.** We next proceed to the construction of the representation with  $\alpha_1 = t_1$ ,  $\alpha_2 = t_2$ ,  $t_1 + t_2 = 1$ . Our construction has lots of similarities with *gruppoid construction* of [VK81] for the realization of factor representation of the infinite symmetric group  $S(\infty)$ .

Let  $V$  be the infinite-dimensional linear space over  $\mathbb{F}_q$  with basis  $e_1, e_2, \dots$ . Denote  $V_i = \langle e_1, \dots, e_i \rangle$ . In what follows we use an infinite-dimensional analogue of the well-known decomposition of grassmanian into Schubert cells.

**Definition 5.4.** For a subspace  $X$  of  $V$  with  $d_i = \dim(X \cap V_i)$ , the symbol of  $X$  is the 0 – 1 sequence  $d_i - d_{i-1}$ :

$$Sym(X) := (d_1 - d_0, d_2 - d_1, \dots).$$

**Definition 5.5.** For a 0 – 1 sequence  $x$  let Schubert cell of  $x$  denote the set of all subspaces with symbol  $x$ :

$$Sch(x) = \{X \subset V \mid Sym(X) = x\}.$$

We fix a distinguished *coordinate* subspace in  $Sch(x)$  which is

$$C(x) = \langle e_i \mid x_i = 1 \rangle.$$

In the same way if  $X$  is a subspace of  $V_n$ , then its  $n$ -dimensional symbol  $Sym^n(X)$  is the 0 – 1 sequence  $(d_1 - d_0, \dots, d_n - d_{n-1})$  of length  $n$ . For a 0 – 1 sequence  $(x_1, \dots, x_n)$  we define a finite Schubert cell

$$Sch^n(x) = \{X \subset V_n \mid Sym^n(X) = x\}.$$

By a simple linear algebra we have

$$(5.1) \quad |Sch^n(x)| = q^{\sum_{i=1}^n (ix_i) - m(m+1)/2},$$

where  $m = \sum_{i=1}^n x_i$ .

**Remark.** Another way to rewrite (5.1) is

$$|Sch^n(x)| = q^{\text{inv}(-x)},$$

where  $\text{inv}(-x)$  is the number of *inversions* in  $-x$ . In other words, it is the number of pairs  $i < j$  such that  $x_i < x_j$ . Similar formula still holds when we pass from grassmanian to more complicated flag varieties. This makes a link to  $q$ -exchangeability and  $\mathbb{GL}(\infty, q)$ -invariant measures on flags of [GO2].

Let  $\nu_x^n$  denote the uniform probability measure on the finite set  $Sch^n(x)$ . Thus, for a subspace  $X$  in  $V_n$  we have

$$\nu_x^n(X) = \begin{cases} q^{m(m+1)/2 - \sum_{i=1}^n (ix_i)}, & X \in Sch^n(x), \\ 0, & \text{otherwise.} \end{cases}$$

For a space  $W$  ( $W$  will be either  $V$  of  $V_n$ ) let  $Gr(W)$  the set of all subspaces of  $W$ . Note that **GLB** naturally acts in  $Gr(V)$ . We equip  $Gr(V)$  with a topology of **GLB**-space (i.e. elementary open neighborhood of a point  $x$  is the image of the action on  $x$  of an open neighborhood of identity element in **GLB**) and corresponding  $\sigma$ -algebra of Borel sets.  $Gr(V)$  is a union of Schubert cells, every cell is a measurable subset of  $Gr(V)$  and is a **B**-orbit. Let  $\pi_x$  be the map:

$$\pi_x : \mathbf{B} \rightarrow Gr(V), \quad \pi(g) = gC(x).$$

Let measure  $\nu_x$  be the image of the Haar measure on **B** with respect to  $\pi_x$ . By its definition  $\nu_x$  is a unique **B**-invariant probability measure supported on  $Sch(x)$ .

Let  $\pi^{(n)}$  be the projection

$$\Pi^{(n)} : Gr(V) \rightarrow Gr(V_n), \quad \pi^{(n)}(X) = X \bigcap V_n,$$

then the image of  $\nu_x$  with respect to the map  $\pi^{(n)}$  is precisely the uniform probability measure  $\nu_{(x_1, \dots, x_n)}^n$  on the finite Schubert cell  $Sch^n((x_1, \dots, x_n)) \subset Gr(V_n)$ .

Let us introduce an important probability measure  $\eta_{t_1, t_2}$  on  $Gr(V)$ . Let  $\phi$  denote the map

$$\phi : \{0, 1\}^\infty \times \mathbf{B} \rightarrow Gr(V), \quad (x, g) \rightarrow gC(x).$$

**Definition 5.6.** The measure  $\eta_{t_1, t_2}$  is the  $\phi$ -pushforward of the product of Bernoulli measure with probability of 1 being  $t_1$ , and Haar measure  $\mu_{\mathbf{B}}$  on **B**. In other words, to get a random element of  $Gr(V)$  distributed according to the measure  $\eta_{t_1, t_2}$  we, first, sample a 0–1 sequence  $x$  from the Bernoulli measure and then take an element of  $Sch(x)$  distributed according to  $\nu_x$ .

We also let  $\eta_{t_1, t_2}^n$  be the  $\Pi^{(n)}$  pushforward of  $\eta_{t_1, t_2}$ . Our definitions imply that for  $X \in Gr(V_n)$  with symbol  $(x_1, \dots, x_n)$  we have

$$(5.2) \quad \eta_{t_1, t_2}^n(X) = q^{m(m+1)/2 - \sum_{i=1}^n (ix_i)} t_1^{\sum_i x_i} t_2^{n - \sum_i x_i}.$$

The following two propositions explain the relation between  $\eta_{t_1, t_2}$  and action of **GLB**.

**Proposition 5.7** (Fundamental cocycle of the action on grassmanian). *The measure  $\eta_{t_1, t_2}$  is quasi-invariant with respect to the action of **GLB**. The cocycle of the action of **GLB** is given by*

$$\frac{\eta_{t_1, t_2}(g \cdot dX)}{\eta_{t_1, t_2}(dX)} = q^{\sum_k k(\text{Sym}(X)_k - \text{Sym}(gX)_k)}.$$

*Proof.* By the definition  $\eta_{t_1, t_2}$  is **B**-invariant. Thus, it remains to consider  $g \in \mathbb{GL}(n, q)$  for arbitrary  $n$ . But then the computation of the cocycle of  $\eta_{t_1, t_2}$  boils down to the computation for  $\eta_{t_1, t_2}^n$  which is straightforward from (5.2).  $\square$

**Proposition 5.8.** *If  $t_1$  and  $t_2$  are nonzero, then there is no finite or  $\sigma$ -finite **GLB**-invariant measure on  $Gr(V)$  equivalent (i.e. with the same sets of measure zero) to  $\eta_{t_1, t_2}$ .*

*Proof.* The classification of all finite **GLB**-invariant measures on  $Gr(V)$  was recently found in [GO2]. It is shown there that every **GLB**-invariant probability measure is supported on the subspaces whose symbol has finitely many 0s or 1s. Thus, finite **GLB**-invariant measure cannot be equivalent to  $\eta_{t_1, t_2}$ .

For a subspace  $Y \in Gr(V_n)$  denote

$$U(Y; n) = \{X \in Gr(V) \mid X \bigcap V_n = Y\}.$$

If  $\varphi$  is not finite, but equivalent to  $\eta_{t_1, t_2}$  then for arbitrary  $k, l$  there exists  $n(k, l)$  and  $Z(k, l) \in Gr(V_{n(k, l)})$  such that the symbol of  $Z(k, l)$  has at least  $k$  zeros and at least  $l$  1s and  $\varphi(U(Z(k, l); n(k, l))) = \infty$ . Observe that *any* set  $U(Y; n)$  contains an image of one of such sets  $Z(k, l)$  with respect to **GLB**-action. We conclude that for a non-finite **GLB**-invariant equivalent to  $\eta_{t_1, t_2}$  measure  $\varphi$ ,  $\varphi(U(Y; n)) = \infty$  for any  $Y$  and  $n$ . Therefore,  $\varphi$  is not  $\sigma$ -finite.  $\square$

In order to get the desired spherical representation we need to introduce a more complicated space. The construction of this space has similarities with analogous construction for infinite symmetric group  $S(\infty)$ , see [VK81], [TV] with some ideas tracing back to the papers of F. J. Murray and J. von Neumann [MN], [N].

Let

$$Gr^2(V) = \{(X, Y) \in Gr(V) \times Gr(V) \mid X = gY, \text{ for some } g \in \mathbf{GLB}\}.$$

We equip the set  $Gr^2(V)$  with a topology, the elementary open neighborhoods of a point  $(X, Y)$  are indexed by numbers  $n = 0, 1, 2, \dots$  and

$$U^n(X, Y) = \left\{ (Z, W) \in Gr^2(V) \mid Z \bigcap V_n = X \bigcap V_n, W \bigcap V_n = Y \bigcap V_n, \right. \\ \left. \dim(Z \bigcap V_k) = \dim(W \bigcap V_k), \text{ for } k \geq n \right\}.$$

Note that  $U^n(X, Y)$  actually depends only on  $X \cap V_n, Y \cap V_n$  and this set is empty unless  $\dim(X \cap V_n) = \dim(Y \cap V_n)$ . In this topology  $Gr^2(V)$  is locally compact. The group **GLB**  $\times$  **GLB** naturally acts in  $Gr^2(V)$  and the action is continuous in the introduced topology.

Now we introduce a measure  $\rho_{t_1, t_2}$  on  $Gr^2(V)$  which is quasiinvariant with respect to the action of **GLB**  $\times$  **GLB**.

Let  $x, y$  be two infinite 0–1 sequences. We write  $x \sim y$  if there exists  $N$  such that  $x_n = y_n$  for  $n > N$  and  $\sum_{n=1}^N x_n = \sum_{n=1}^N y_n$ . Note that  $(X_1, X_2) \in Gr(V) \times Gr(V)$  belongs to  $Gr^2(V)$  is and only if  $\text{Sym}(X_1) \sim \text{Sym}(X_2)$ .

Denote  $\mathcal{T}$  the set of pairs  $(x, y)$  of infinite 0–1 sequences such that  $x \sim y$ .  $\mathcal{T}$  is equipped with sigma algebra spanned by the sets  $A_{i_1, \dots, i_n}^{j_1, \dots, j_n}$ , where  $i_1, \dots, i_n$  and  $j_1, \dots, j_n$  are two 0–1 sequences such that  $\sum i_k = \sum j_k$ .  
 $A_{i_1, \dots, i_n}^{j_1, \dots, j_n} = \{(x, y) \in \mathcal{T} \mid x_1 = i_1, \dots, x_n = i_n, y_1 = j_1, \dots, y_n = j_n, y_k = x_k, \text{ for } k > n\}$ .  
 We define the measure  $R_{t_1, t_2}$  by

$$R_{t_1, t_2}(A_{i_1, \dots, i_n}^{j_1, \dots, j_n}) = t_1^{\sum_k i_k} t_2^{n - \sum_k i_k}.$$

Let  $\psi$  be the map

$$\psi : \mathcal{T} \times \mathbf{B} \times \mathbf{B} \rightarrow \text{Gr}^2(V), \quad \psi((x, y, g, h)) = (gC(x), hC(y)),$$

and let  $\rho_{t_1, t_2}$  be the push-forward of the measure  $R_{t_1, t_2} \otimes \mu_{\mathbf{B}} \otimes \mu_{\mathbf{B}}$  with respect to  $\psi$ .

The following proposition gives a more direct description of the measure  $\rho_{t_1, t_2}$ .

**Proposition 5.9.** *Let  $n = 0, 1, 2, \dots$  and  $(X, Y) \in \text{Gr}^2(V)$ . Suppose that  $\dim(X \cap V_k) = \dim(Y \cap V_k)$  for  $k \geq n$  and denote*

$$m := \dim(X \cap V_n) = \dim(Y \cap V_n) = \sum_{k=1}^n \text{Sym}(X)_k.$$

We have

$$\rho_{t_1, t_2}(U^n(X, Y)) = q^{m(m+1) - \sum_k k(\text{Sym}(X)_k + \text{Sym}(Y)_k)} t_1^m t_2^{n-m}.$$

*Proof.* Observe that if  $(X', Y')$  are such that  $\text{Sym}(X') = \text{Sym}(X)$  and  $\text{Sym}(Y') = \text{Sym}(Y)$ , then there exist  $(g, h) \in \mathbf{B} \times \mathbf{B}$  such that  $X' = gX$ ,  $Y' = hY$ . Therefore

$$U^n(X', Y') = (g, h)U^n(X, Y),$$

hence, by the definition of the measure,

$$\rho_{t_1, t_2}(U^n(X, Y)) = \rho_{t_1, t_2}(U^n(X', Y')).$$

Now note that

$$(5.3) \quad \psi(A_{\text{Sym}(X)_1, \dots, \text{Sym}(X)_n}^{\text{Sym}(Y)_1, \dots, \text{Sym}(Y)_n} \times \mathbf{B} \times \mathbf{B}) = \bigcup_{(Z_i, W_i)} U^n(Z, W),$$

where  $Z_i$  goes over  $q^{\sum_k k(\text{Sym}(X)_k - m(m+1)/2)}$  subspaces of  $V$  such that  $\text{Sym}(Z_i) = \text{Sym}(X)$  and  $Z_i \cap V_n$  are pairwise distinct;  $W$  goes over  $q^{\sum_k k(\text{Sym}(Y)_k - m(m+1)/2)}$  subspaces of  $V$  such that  $\text{Sym}(W_i) = \text{Sym}(Y)$  and  $W_i \cap V_n$  are pairwise distinct. Evaluating  $\rho_{t_1, t_2}$  of both sides of (5.3) we get the desired formulas.  $\square$

**Corollary 5.10.** *The measure  $\rho_{t_1, t_2}$  is quasi-invariant with respect to the action of  $\mathbf{GLB} \times \mathbf{GLB}$ . The corresponding cocycle is given by*

$$\frac{\rho_{t_1, t_2}((g, h) \cdot d(X, Y))}{\rho_{t_1, t_2}(d(X, Y))} = q^{\sum_k k(\text{Sym}(X)_k - \text{Sym}(gX)_k) + \sum_k k(\text{Sym}(Y)_k - \text{Sym}(hY)_k)}.$$

*Proof.* This follows from Proposition 5.9 and the fact that if  $(g, h) \in \mathbf{GLB} \times \mathbf{GLB}$  and  $n$  is large enough integer, then

$$(g, h)U^n(X, Y) = U^n(gX, hY).$$

$\square$



**Remark.** It is easy to replace the measure  $\rho_{t_1, t_2}$  with an equivalent one  $\widehat{\rho}_{t_1, t_2}$  which would be invariant with respect to the action of the subgroup  $\mathbf{GLB} \times \{e\} \subset \mathbf{GLB} \times \mathbf{GLB}$ . However, it is not possible to achieve the invariance with respect to the whole group  $\mathbf{GLB} \times \mathbf{GLB}$ .

Further, let  $\pi_{t_1, t_2}$  denote the usual unitary representation of  $\mathbf{GLB} \times \mathbf{GLB}$  in the  $L_2(Gr^2(V), \rho_{t_1, t_2})$ . In other words, for  $f \in L_2(Gr^2(V), \rho_{t_1, t_2})$  and  $(g, h) \in \mathbf{GLB} \times \mathbf{GLB}$  we have

$$[\pi_{t_1, t_2}(g, h)f](X, Y) = f(g^{-1}X, h^{-1}Y) \sqrt{\frac{\rho_{t_1, t_2}((g^{-1}, h^{-1}) \cdot d(X, Y))}{\rho_{t_1, t_2}(d(X, Y))}}.$$

Let  $C^0(Gr^2(V))$  denote the space of continuous functions on  $Gr^2(V)$  with compact support equipped with supremum-norm. We have natural inclusions

$$C^0(Gr^2(V)) \subset L_2(Gr^2(V), \rho_{t_1, t_2}) \subset (C^0(Gr^2(V)))^*.$$

Consider the unitary representation of  $\mathbf{GLB} \times \mathbf{GLB}$  dual to the restriction of  $\pi_{t_1, t_2}$  on  $C^0(Gr^2(V))$ . Somewhat abusing the notations we will use the same symbol  $\pi_{t_1, t_2}$  for this representation.

Let  $v_{t_1, t_2} \in (C^0(Gr^2(V)))^*$  denote the linear functional:

$$v_{t_1, t_2} : C^0(Gr^2(V)) \rightarrow \mathbb{C}, \quad v_{t_1, t_2}(f) = \int_{Gr(V)} f(X, X) \eta_{t_1, t_2}(dX).$$

Further, let  $Sp(v_{t_1, t_2})$  denote the  $\mathbf{GLB} \times \mathbf{GLB}$  cyclic span of the linear functional  $v_{t_1, t_2}$  and let  $\widehat{\pi}_{t_1, t_2}$  denote the restriction of  $\pi_{t_1, t_2}$  on  $L_2$ -closure of  $Sp(v_{t_1, t_2}) \cap C^0(Gr^2(V))$ .

**Theorem 5.11.** *The triplet  $(\widehat{\pi}_{t_1, t_2}, C^0(Gr^2(V)) \cap Sp(v_{t_1, t_2}), v_{t_1, t_2})$  is a generalized spherical representation. Its spherical function gives the extreme unipotent trace of  $\mathcal{A}(\mathbf{GLB})$  with parameters  $\alpha_1 = t_1$ ,  $\alpha_2 = t_2$ . The restriction of this representations on the first component is von Neumann factor representation of type  $II_\infty$ .*

*Proof.* Let us check the 6 properties of a generalized spherical representation.

- (1) The natural map  $i : C^0(Gr^2(V)) \rightarrow L_2(Gr^2(V), \rho_{t_1, t_2})$  is, indeed, a continuous inclusion. This follows from our choice of topology.
- (2) Since for any element  $(g, h) \in \mathbf{GLB} \times \mathbf{GLB}$  and any elementary open neighborhood  $U^n(X, Y)$  (with large enough  $n$ ) we have  $(g, h)U^n(X, Y) = U^n(gX, gY)$ , thus, the action of  $\mathbf{GLB} \times \mathbf{GLB}$  maps open sets to open sets and, therefore, preserves the space of continuous functions with compact support.
- (3) It suffices to prove that  $(I_{e(n)}, e)v_{t_1, t_2}$  is a continuous function with compact support. Let  $\mathbf{BI}_n$  denote the subgroup  $\text{Cyl}_{e(n)}^{\mathbf{GLB}} \subset \mathbf{B}$ . By the definition

$$\begin{aligned} ((I_{e(n)}, e)v_{t_1, t_2}, f) &= \int_{X \in Gr(V)} \int_{g \in \mathbf{BI}_n} ((g, e) \cdot f)(X, X) \mu(dg) \eta_{t_1, t_2}(dX) \\ &= \int_{X \in Gr(V)} \int_{g \in \mathbf{BI}_n} \sqrt{\frac{(g^{-1}, e) \rho_{t_1, t_2}(X, X)}{\rho_{t_1, t_2}}} f(g^{-1}X, X) \mu(dg) \eta_{t_1, t_2}(dX) \\ &= \int_{X \in Gr(V)} \int_{g \in \mathbf{BI}_n} f(g^{-1}X, X) \mu(dg) \eta_{t_1, t_2}(dX) \end{aligned}$$

For  $X, Y \in Gr(V)$  let  $\mathcal{U}^k(X, Y)$  denote the indicator function of the set  $U^k(X, Y)$ . To analyze the last integral we set  $f = \mathcal{U}^k(I, J)$ ,  $k > n$ ,  $I, J \in V^k$  and compute  $\int_{g \in \mathbf{BI}_n} f(g^{-1}X, X)\mu(dg)$ . Observe that if  $X \cap V_k \neq J$  or  $X \cap V_n \neq I \cap V_n$  or  $\dim(X \cap V_\ell) \neq \dim(I \cap V_\ell)$  for some  $n \leq \ell \leq k$ , then  $(g^{-1}X, X)$  does not belong to  $U^k(I, J)$  and the integral vanishes. Otherwise, it is equal to

$$\int_{g \in \mathbf{BI}_n} f(g^{-1}X, X)\mu(dg) = \mu(\mathbf{BI}_n)/M,$$

where  $M$  is the number of  $Z \in Gr(V_k)$  such that  $Z \cap V_n = I \cap V_n$  and  $\dim(Z \cap V_\ell) = \dim(I \cap V_\ell)$  for all  $n \leq \ell \leq k$ . We have

$$1/M = q^{\dim(I)(\dim(I)+1)/2 - \dim(I \cap V_n)(\dim(I \cap V_n)+1)/2 - \sum_{\ell=n+1}^k \ell \text{Sym}(I)_\ell}.$$

Therefore, the double integral equals

$$\begin{aligned} (5.4) \quad & \int_{X \in Gr(V)} \int_{g \in \mathbf{BI}_n} f(g^{-1}X, X)\mu(dg)\eta_{t_1, t_2}(dX) \\ &= \mu(\mathbf{BI}_n)q^{\dim(I)(\dim(I)+1)/2 - \dim(I \cap V_n)(\dim(I \cap V_n)+1)/2 - \sum_{\ell=n+1}^k \ell \text{Sym}(I)_\ell} \eta_{t_1, t_2}^k(J) \\ & \quad \text{if } I \cap V_n = J \cap V_n \text{ and } \dim(I \cap V_\ell) = \dim(J \cap V_\ell) \text{ for } n \leq \ell \leq k, \\ & \quad \text{otherwise the double integral vanishes. (5.4) together with formulas for} \\ & \quad \rho_{t_1, t_2}(U^k(I, J)) \text{ and } \eta_{t_1, t_2}^k(J) \text{ imply} \end{aligned}$$

$$\begin{aligned} (5.5) \quad & \int_{X \in Gr(V)} \int_{g \in \mathbf{BI}_n} f(g^{-1}X, X)\mu(dg)\eta_{t_1, t_2}(dX) \\ &= \int_{Gr^2(V)} v^n(X, Y)f(X, Y)\rho_{t_1, t_2}(d(X, Y)), \end{aligned}$$

where

$$v_{t_1, t_2}^n(X, Y) = \begin{cases} \mu(\mathbf{BI}_n)q^{-\dim(X \cap V_n)(\dim(X \cap V_n)+1)/2 + \sum_{\ell=1}^n \ell \text{Sym}(X)_\ell}, & \text{if } (X, Y) \in L_n, \\ 0, & \text{otherwise,} \end{cases}$$

$$L_n = \{(X, Y) \in Gr^2(V) \mid$$

$$X \cap V_n = Y \cap V_n, \dim(X \cap V_k) = \dim(Y \cap V_k) \text{ for } k > n\}.$$

Since the linear span of the functions  $\mathcal{U}^k(I, J)$  (with various  $k$ ) is dense in  $L_2(Gr^2(V), \rho_{t_1, t_2})$ , the equality (5.5) holds for a general function  $f(X, Y)$ . Thus,  $(I_{e(n)}, e)v_{t_1, t_2} = v_{t_1, t_2}^n$  is, as desired, a continuous function with compact support.

- (4) Since we work in the span of  $v_{t_1, t_2}$ , there is nothing to check here.
- (5) Let us check that  $(g, g)v_{t_1, t_2} = v_{t_1, t_2}$  for any  $g \in \mathbf{GLB}$ . Indeed, since  $\pi_{t_1, t_2}$  is unitary representation,

$$\begin{aligned} ((g, g)v_{t_1, t_2}, f) &= (v_{t_1, t_2}, (g^{-1}, g^{-1})f) \\ &= \int_{Gr(V)} f(gX, gX) \sqrt{\frac{(g, g)\rho_{t_1, t_2}}{\rho_{t_1, t_2}}}(X, X)\eta_{t_1, t_2}(dX) \\ & \quad \stackrel{Y=gX}{=} \int_{Gr(V)} f(Y, Y)\eta_{t_1, t_2}(dY) = (v_{t_1, t_2}, f) \end{aligned}$$

(6) We have already shown that  $I_{\mathbf{B}} v_{t_1, t_2} = v_{t_1, t_2}^0$ . Then

$$(I_{\mathbf{B}} v_{t_1, t_2}, v_{t_1, t_2}) = \int_{Gr(V)} \eta_{t_1, t_2}(dX) = 1.$$

Now we compute the trace of this representation.

For  $g \in \mathbb{GL}(n, q)$  we have

$$\begin{aligned} (\pi_{t_1, t_2}(I_g, e) v_{t_1, t_2}, v_{t_1, t_2}) &= (\pi_{t_1, t_2}(g \cdot I_{e(n)}, e) v_{t_1, t_2}, v_{t_1, t_2}) = (\pi_{t_1, t_2}(g, e) v^n, v) \\ &= \mu(\mathbf{BI}_n) \sum_{X \in Gr(V^n)} (\pi_{t_1, t_2}(g, e) q^{-\dim(X)(\dim(X)+1)/2 + \sum_k k(\text{Sym}(X)_k)} \mathcal{U}^n(X, X), v) \end{aligned}$$

By the definition for  $X, Y \in Gr(V_n)$  we have

$$(\mathcal{U}^n(X, Y), v) = \begin{cases} q^{m(m+1)/2 - \sum_k k(\text{Sym}(X)_k)} t_1^m t_2^{n-m}, & X = Y, \\ 0, & \text{otherwise,} \end{cases}$$

where  $m = \dim(X)$ . It follows that

$$(5.6) \quad (\pi_{t_1, t_2}(I_g, e) v_{t_1, t_2}, v_{t_1, t_2}) = \mu(\mathbf{BI}_n) \sum_{X \in Gr(V^n): gX=X} t_1^{\dim(X)} t_2^{n-\dim(X)}.$$

Our next aim is to decompose the function  $\chi(g) := (\pi_{t_1, t_2}(I_g, e) v, v)$  into the sum of matrix traces of irreducible representations of  $\mathbb{GL}(n, q)$ . We rewrite (5.6) as

$$\chi(g) = \sum_{m=0}^n t_1^n t_2^{n-m} \psi_m(g)$$

with

$$\psi_m(g) = \mu(\mathbf{BI}_n) \# \{X \in Gr(V^n) : \dim(X) = m, gX = X\}$$

Let  $\Psi_m$  be the natural representation of  $\mathbb{GL}(n, q)$  in the space of functions on the set of all subspaces of  $V_n$  of dimension  $m$ . If we view  $\Psi_m$  as a representation of  $\mathcal{A}(\mathbf{GLB})_n$ , then its matrix trace  $\text{Trace}(\Psi_m(I_g))$  is precisely  $\psi_m(g)$ ; the prefactor  $\mu(\mathbf{BI}_n)$  arises from the identification of  $\mathcal{A}(\mathbf{GLB})$  and the group algebra of  $\mathbb{GL}(n, q)$  (see Proposition 2.5).

The decomposition of  $\Psi_m$  into irreducible representations is well known (see e.g. [St]). We have

$$(5.7) \quad \Psi_m = \bigoplus_{\lambda} K_{(n-m, m), \lambda} \psi^\lambda,$$

where  $\psi^\lambda$  is the irreducible unipotent representation of  $\mathbb{GL}(n, q)$  indexed by the Young diagram with  $n$  boxes  $\lambda$  and  $K_{(n-m, m), \lambda}$  is the *Kostka number*. These numbers do not depend on  $q$  and coincide with similar coefficients for the decomposition of the representation of symmetric group  $\mathfrak{S}(n)$  in the space of functions on the set of all  $m$ -element subsets of the set  $\{1, 2, \dots, n\}$ . It is convenient for us to use yet another definition related to the symmetric functions:

$$h_m h_{n-m} = \sum_{\lambda} K_{(n-m, m), \lambda} s_\lambda,$$

where  $h_m$  is the complete symmetric function and  $s_\lambda$  is the Schur function. The last formula can be shown to be equivalent to the definition of Kostka numbers through (3.12).

(5.7) implies that

$$\chi = \sum_{\lambda} \left( \sum_m K_{(n-m,m),\lambda} t_1^m t_2^{n-m} \right) \chi^{\lambda},$$

where  $\chi^{\lambda}$  is the conventional character (matrix trace) of the unipotent representation indexed by the Young diagram with  $n$  boxes  $\lambda$ .

Next, observe that

$$\sum_m K_{(n-m,m),\lambda} t_1^m t_2^{n-m} = \sum_{m \leq n/2} K_{(n-m,m),\lambda} m_{(n-m,m)}(t_1, t_2) = s_{\lambda}(t_1, t_2),$$

where  $m_{\mu}$  is the monomial symmetric function indexed by the Young diagram  $\mu$  and for a symmetric function  $f(x_1, x_2, \dots)$  the notation  $f(t_1, t_2)$  means the specialization  $f(t_1, t_2, 0, 0, \dots)$ .

We arrive at the final formula

$$\chi = \sum_{\lambda} s_{\lambda}(t_1, t_2) \chi^{\lambda},$$

which coincides with the decomposition of the extreme unipotent trace of  $\mathcal{A}(\mathbf{GLB})$  with parameters  $\alpha_1 = t_1, \alpha_2 = t_2$  given in Theorem 2.26.  $\square$

**5.4. Representations related to spaces of flags.** The results of Section 5.3 can be generalized to give a construction for the representations of **GLB** corresponding to the extreme unipotent representation with arbitrary sequence of parameters  $\alpha_i$ .

Suppose that we have  $r$  non-zero parameters  $\alpha_i$ :

$$\alpha_1 = t_1, \alpha_2 = t_2, \dots, \alpha_r = t_r,$$

with  $r$  being either finite or  $r = +\infty$ .

Let  $Fl_r(V)$  denote the space of all length  $r-1$  *decreasing* flags in  $V$ , i.e.

$$Fl_r(V) = \{X_1 \supseteq X_2 \supseteq \dots \supseteq X_{r-1} \mid X_i \in Gr(V)\}.$$

In particular,  $Fl_2(V) = Gr(V)$ . Note that, in principle, we allow non-strict inclusions in the above definition, e.g.  $X_1$  might be equal to  $X_2$ . However, with respect to the measures we use, the inclusions turn out to be almost surely strict. If  $r = \infty$ , then we also demand that  $\bigcap_i X_i = \emptyset$ .

The group **GLB** naturally acts in  $Fl_r(V)$  and, similarly to the grassmanian case, we define:

$$Fl_r^2(V) = \{(F, H) \in Fl_r(V) \times Fl_r(V) \mid \exists g \in \mathbf{GLB} : gF = H\}.$$

For a flag  $F \in Fl_r(V)$  let  $F^{(i)}$ ,  $i = 1, \dots, r-1$  denote its subspaces, i.e.  $F = F^{(1)} \supseteq \dots \supseteq F^{(r-1)}$ . The symbol  $Sym(F)$  of the flag  $F \in Fl_k(V)$  is defined as the coordinate-wise sum of the symbols of  $F^{(i)}$ :

$$Sym(F) = \left( \sum_{i=1}^{r-1} Sym(F^{(i)})_1, \sum_{i=1}^{r-1} Sym(F^{(i)})_2, \dots \right).$$

Note that this sum is well-defined even for  $r = \infty$  as follows from the condition  $\bigcap_i X_i = \emptyset$ .

Let  $\mathcal{N}_r$  denote the set  $\{0, 1, \dots, r-1\}$ . For a sequence  $f \in \mathcal{N}_r^{\infty}$  let  $Sch(f)$  denote the set of all flags in  $Fl_r(V)$  with symbol  $f$ .  $Sch(f)$  has a distinguished coordinate flag, which we denote  $C(f)$ .  $Sch(f)$  is a **B**-orbit and, thus, has a unique **B**-invariant probability measure.

Next, we define the map  $\phi_r$ :

$$\phi_r : \mathcal{N}_r^\infty \times \mathbf{B} \rightarrow Fl_r(V), \quad \phi_r(x, g) = gC(x).$$

Let  $\eta_{t_1, \dots, t_r}$  be the  $\phi_r$ -pushforward of the product of Bernoulli measure  $P$  on  $\mathcal{N}_r^\infty$  with  $Prob(k) = t_k$  and Haar measure on  $\mathbf{B}$ .

Let  $\mathcal{T}_r$  denote the set of pairs of sequences  $(x, y) \in \mathcal{N}_r^\infty \times \mathcal{N}_r^\infty$  such that  $x$  is a finite permutation of  $y$ .

$\mathcal{T}_r$  is equipped with sigma algebra spanned by the sets  $A_{i_1, \dots, i_n}^{j_1, \dots, j_n}$ , where  $i_1, \dots, i_n$  and  $j_1, \dots, j_n$  are two sequences from  $\mathcal{N}_r^n$  which are permutations of each other

$$A_{i_1, \dots, i_n}^{j_1, \dots, j_n} = \{(x, y) \in \mathcal{T}_r \mid x_1 = i_1, \dots, x_n = i_n, y_1 = j_1, \dots, y_n = j_n, y_k = x_k, \text{ for } k > n\}.$$

Define the measure  $R_{t_1, \dots, t_r}$  on  $\mathcal{T}_r$  setting

$$R_{t_1, \dots, t_r}(A_{i_1, \dots, i_n}^{j_1, \dots, j_n}) = \prod_{\ell=1}^n t_{i_\ell}.$$

Let  $\psi_r$  be the map

$$\psi_r : \mathcal{T}_r \times \mathbf{B} \times \mathbf{B} \rightarrow Fl_r^2(V), \quad \psi_r((x, y), g, h) = (gC(x), hC(y)).$$

We define the measure  $\rho_{t_1, \dots, t_r}$  on  $Fl_r(V)$  as the  $\psi_r$ -pushforward of the measure  $R_{t_1, \dots, t_r} \otimes \mu_{\mathbf{B}} \otimes \mu_{\mathbf{B}}$ .

Similarly to Proposition 5.7 and Corollary 5.10 one proves that  $\eta_{t_1, \dots, t_r}$  is **GLB**-quasiinvariant and  $\rho_{t_1, \dots, t_r}$  is **GLB**  $\times$  **GLB**-quasiinvariant.

Therefore there is a natural unitary representation  $\pi_{t_1, \dots, t_r}$  of **GLB**  $\times$  **GLB** in  $L_2(Fl_r(V), \rho_{t_1, \dots, t_r})$ .

Similarly, to  $Gr^2(V)$  we define a topological structure of  $Fl_r(V)$  and consider the space  $C^0(Fl_r^2(V))$  of continuous functions with compact support. Let  $v_{t_1, \dots, t_r} \in C^0(Fl_r^2(V))^*$  denote the following linear functional

$$v_{t_1, \dots, t_r} : C^0(Fl_r^2(V)) \rightarrow \mathbb{C}, \quad v_{t_1, \dots, t_r}(f) = \int_{Fl_r(V)} f(X, X) \eta_{t_1, \dots, t_r}(dX).$$

We further set  $\hat{\pi}_{t_1, \dots, t_r}$  to be the restriction of  $\pi_{t_1, \dots, t_r}$  on the  $L_2$ -closure of the intersection of  $C^0(Fl_r^2(V))$  and cyclic span  $Sp(v_{t_1, \dots, t_r})$  of  $v_{t_1, \dots, t_r}$ .

**Theorem 5.12.** *The triplet  $(\hat{\pi}_{t_1, \dots, t_r}, C^0(Fl_r^2(V)) \cap Sp(v_{t_1, \dots, t_r}), v_{t_1, \dots, t_r})$  is a generalized spherical representation. Its spherical function gives the extreme unipotent trace of  $\mathcal{A}(\mathbf{GLB})$  with parameters  $\alpha_1 = t_1, \dots, \alpha_r = t_r$ . The restriction of this representations on the first component is von Neumann factor representation of type  $II_\infty$ .*

The proof repeats that of Theorem 5.11.

## 6. BIREGULAR REPRESENTATION OF **GLB**

Recall that **GLB** is a locally compact group with biinvariant Haar measure  $\mu_{\mathbf{GLB}}$  and consider the Hilbert space  $H = L_2(\mathbf{GLB}, \mu_{\mathbf{GLB}})$ . Let  $\pi_{Reg}$  denote the natural representation of **(GLB**  $\times$  **GLB)** in  $H$  by left and right translations. Let  $H_1 \subset H$  be the subspace  $C[\mathbf{GLB}]$  of all continuous functions on **GLB** and let  $\delta_e$  denote the delta-function at the identity element of **GLB**:

$$\delta_e(f) = f(e), \quad f \in H_1.$$

**Theorem 6.1** (On the structure of biregular representation). *The triplet  $(\pi_{Reg}, C[\mathbf{GLB}], \delta_e)$  is a generalized spherical representation of  $\mathbf{GLB}$ . Its spherical function  $\chi$  has the following decomposition into extreme traces of  $\mathcal{A}(\mathbf{GLB})$ :*

$$\chi = \sum_{f \in \mathcal{CY}'} C(f) \chi^{1^{(q)}, 0, f},$$

where  $1^{(q)}$  means the geometric series  $((1 - q^{-1}), (1 - q^{-1})q^{-1}, (1 - q^{-1})q^{-2}, \dots)$  and

$$C(f) = (q - 1)^{|f|} \prod_{c \in \mathcal{C}_d} \frac{q^{dn(s(c)')}}{\prod_{\square \in s(c)} (q^{dh(\square)} - 1)}.$$

*Proof.* Observe that  $\pi_{Reg}(I_{e(n)}, e)\delta_e = I_{e(n)}$ . Therefore, for  $g \in \mathbb{GL}(n, q)$  we have

$$\pi_{Reg}(I_g, e)\delta_e = \begin{cases} 1, & g = e(n), \\ 0, & \text{otherwise.} \end{cases}$$

It follows that the restriction of  $\chi$  to  $\mathcal{A}(\mathbf{GLB})_n$  (under the identification  $\mathcal{A}(\mathbf{GLB})_n \simeq \mathbb{C}(\mathbb{GL}(n, q))$ ) is the character of the regular representation of  $\mathbb{GL}(n, q)$  multiplied by the constant

$$\frac{(q - 1)^n q^{n(n-1)/2}}{|\mathbb{GL}(n, q)|} = \prod_{i=1}^n \frac{q - 1}{q^i - 1}.$$

Using the well-known decomposition of the regular representation of a finite group into irreducibles we get

$$\chi|_{\mathcal{A}(\mathbf{GLB})_n} = \prod_{i=1}^n \frac{q - 1}{q^i - 1} \cdot \sum_{s \in \mathcal{CY}_n} \dim_q(s) \chi^g,$$

where  $\dim_q(h)$  is the dimension of the irreducible representation of  $\mathbb{GL}(n, q)$  indexed by  $h$  which can be computed using the following  $q$ -analogue of the hook formula:

$$\dim_q(s) = (q^n - 1) \dots (q - 1) \prod_{d \geq 1} \prod_{c \in \mathcal{C}_d} \frac{q^{dn(s(c)')}}{\prod_{\square \in s(c)} (q^{dh(\square)} - 1)}$$

Extracting the factor with  $c = "x - 1"$  and using the identity (see e.g. [M, Chapter I, Section 3, Exercise 2])

$$s_\lambda \left( 1 - q^{-1}, q^{-1}(1 - q^{-1}), q^{-2}(1 - q^{-1}), \dots \right) = (q - 1)^{|\lambda|} \frac{q^{n(\lambda)'}}{\prod_{\square \in \lambda} (q^{h(\square)} - 1)}$$

we arrive at  
(6.1)

$$\chi|_{\mathcal{A}(\mathbf{GLB})_n} = \sum_{f \in \mathcal{CY}'} C(f) \sum_{\lambda \in \mathbb{Y}_{n-|f|}} \chi^{f+E_1(\lambda)} s_\lambda \left( 1 - q^{-1}, q^{-1}(1 - q^{-1}), q^{-2}(1 - q^{-1}), \dots \right),$$

with

$$C(f) = (q - 1)^{|f|} \prod_{c \in \mathcal{C}_d} \frac{q^{dn(s(c)')}}{\prod_{\square \in s(c)} (q^{dh(\square)} - 1)}.$$

It remains to compare (6.1) with (2.3).  $\square$

7. APPENDIX: **GLU**

There is another distinguished infinite-dimensional group over finite field, which is a group of almost uni-uppertriangular matrices.

**Definition 7.1.** **GLU** is a subgroup of **GLB** defined through

$$\mathbf{GLU} = \{[X_{ij}] \in \mathbf{GLB} : X_{ii} = 1 \text{ for large enough } i\}.$$

The whole theory for **GLU** is very much parallel to that of **GLB**.

The group **GLU** is an inductive limit of groups  $\mathbf{GLU}_n$ :  $\mathbf{GLU} = \bigcup_{n=0}^{\infty} \mathbf{GLU}_n$ .

$$\mathbf{GLU}_n = \{[X_{ij}] \in \mathbf{GLU} \mid X_{ij} = 0 \text{ if both } i > j \text{ and } i > n; \quad X_{ii} = 1 \text{ for } i > n\},$$

in particular,  $\mathbf{GLU}_0 = \mathbf{U} \subset \mathbf{GLU}$  is the subgroup of all unipotent upper-triangular matrices.

Each  $\mathbf{GLU}_n$  is compact group (with topology of pointwise convergence of matrix elements). **GLU** as an inductive limit of  $\mathbf{GLU}_n$  is a locally compact topological group. Let  $\mu^{\mathbf{GLU}}$  denote the biinvariant Haar measure on **GLU** normalized by the condition  $\mu^{\mathbf{GLU}}(\mathbf{U}) = 1$ .

The space  $L_1(\mathbf{GLU}, \mu^{\mathbf{GLU}})$  is a Banach involutive algebra with multiplication given by the convolution.

**Definition 7.2.**  $\mathcal{A}(\mathbf{GLU}) \subset L_1(\mathbf{GLU}, \mu^{\mathbf{GLU}})$  is defined as the subalgebra formed by all locally constant functions with compact support. In other words, a function  $f(X)$  belongs to  $\mathcal{A}(\mathbf{GLU})$  if there exists  $n$  and a function  $f_n : \mathbb{GL}(n, q) \rightarrow \mathbb{C}$  such that:

$$f(X) = \begin{cases} f_n(X^{(n)}), & \text{if } X \in \mathbf{GLU}_n, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly,  $\mathcal{A}(\mathbf{GLU})$  is dense in  $L_1(\mathbf{GLU}, \mu^{\mathbf{GLU}})$ . Note that  $\mathcal{A}(\mathbf{GLU})$  does not have a unit element.

As above, we call a continuous function  $\chi : \mathcal{A}(\mathbf{GLU}) \rightarrow \mathbb{C}$  a trace of  $\mathcal{A}(\mathbf{GLU})$  if

- (1)  $\chi$  is central, i.e.  $\chi(WU) = \chi(UW)$ ,
- (2)  $\chi$  is positive definite, i.e.  $\chi(W^*W) \geq 0$  for any  $W \in \mathcal{A}(\mathbf{GLU})$ ,

**Remark.** It is impossible to normalize the traces, i.e. for any  $a \in \mathcal{A}(\mathbf{GLU})$  there exists a trace  $\chi$  such that  $\chi(a) = 0$ .

For any matrix  $g \in \mathbb{GL}(n, q)$  let  $I_g^{\mathbf{GLU}} \in \mathcal{A}(\mathbf{GLU})$  denote the function

$$I_g^{\mathbf{GLU}}(X) = \begin{cases} 1, & \text{if } X \in \mathbf{GLU}_n \text{ and } X^{(n)} = g, \\ 0, & \text{otherwise.} \end{cases}$$

Let  $e(n)$  denote the identity element of  $\mathbb{GL}(n, q)$ . Then

$$I_g^{\mathbf{GLU}}(X) = I_{e(n)}^{\mathbf{GLU}}(Xg^{-1}) = g \cdot I_{e(n)}^{\mathbf{GLU}}.$$

Denote  $\mathcal{A}(\mathbf{GLU})_n = \langle I_g^{\mathbf{GLU}} \mid g \in \mathbb{GL}(n, q) \rangle$ . The following proposition is straightforward

**Proposition 7.3.**  $\mathcal{A}(\mathbf{GLU})_n$  is a subalgebra of  $\mathcal{A}(\mathbf{GLU})$  isomorphic to the conventional group algebra  $\mathbb{C}(\mathbb{GL}(n, q))$ . If  $e_g$  denotes the natural basis of  $\mathbb{C}(\mathbb{GL}(n, q))$  then the isomorphism is given  $e_g \rightarrow q^{n(n-1)/2} I_g^{\mathbf{GLU}}$ .



Observe that  $\mathcal{A}(\mathbf{GLU})_n \subset \mathcal{A}(\mathbf{GLU})_{n+1}$ . In the basis  $I_g^{\mathbf{GLU}}$  this inclusion is given by

$$i_n : I_g^{\mathbf{GLU}} \rightarrow \sum_{h \in \text{Ext}^{\mathbf{GLU}}(g)} I_h^{\mathbf{GLU}},$$

where for  $g \in \mathbb{GL}(n, q)$  we have

$$\text{Ext}^{\mathbf{GLU}}(g) = \left\{ [h_{ij}] \in \mathbb{GL}(n+1, q) \mid \begin{array}{l} h^{(n)} = g \text{ and } h_{n,1} = h_{n,2} = \cdots = h_{n,n-1} = 0, \\ h_{n,n} = 1 \end{array} \right\}.$$

The algebra  $\mathcal{A}(\mathbf{GLU})$  can be identified with the inductive limit of algebras  $\mathcal{A}(\mathbf{GLU})_n$ :

$$\mathcal{A}(\mathbf{GLU}) = \varinjlim_{n \rightarrow \infty} \mathcal{A}(\mathbf{GLU})_n = \bigcup_n \mathcal{A}(\mathbf{GLU})_n.$$

Thus,  $\mathcal{A}(\mathbf{GLU})$  is a *locally semisimple* algebra.

**Proposition 7.4.** *The set of all traces of  $\mathcal{A}(\mathbf{GLU})_n$  is a simplicial cone spanned by traces  $\chi^f$ ,  $f \in \mathcal{CY}_n$ . In other words, if  $\chi^n$  is a trace of  $\mathcal{A}(\mathbf{GLU})_n$ , then there exist unique nonnegative coefficients  $c(f)$  such that*

$$\chi^n(\cdot) = \sum_{f \in \mathcal{CY}_n} c(f) \chi^f(\cdot).$$

*Proof.* The proof repeats that of Proposition 2.10 □

**Definition 7.5.** *For two families  $f \in \mathcal{CY}_n$  and  $g \in \mathcal{CY}_{n-1}$  we say that  $g$  precedes  $f$  and write  $g \prec_{\mathbf{GLU}} f$  if there exists  $a \in \mathbb{F}_q^*$  for which*

- (1)  $f("x - a") \setminus g("x - a")$  is one box,
- (2)  $f(u) = g(u)$  for all  $u \neq "x - a"$ .

Note that this definition is *different* from that of  $\prec_{\mathbf{GLB}}$ .

Similarly to Theorem 2.13 one proves the following statement.

**Proposition 7.6.** *Let  $\pi^f$  be the irreducible representation of algebra  $\mathcal{A}(\mathbf{GLU})_n$  (equivalently, of the group  $\mathbb{GL}(n, q)$ ) parameterized by  $f \in \mathcal{CY}_n$  and let  $\chi^f$  be its conventional character (i.e. matrix trace). The restrictions of  $\pi^f$  and  $\chi^f$  to the subalgebra  $\mathcal{A}(\mathbf{GLU})_{n-1}$  admit the following decomposition:*

$$\chi^f|_{\mathcal{A}(\mathbf{GLU})_{n-1}} = \sum_{g \prec_{\mathbf{GLU}} f} \chi^g,$$

equivalently,

$$\pi^f|_{\mathcal{A}(\mathbf{GLU})_{n-1}} = \mathcal{N} \oplus \bigoplus_{g \prec_{\mathbf{GLU}} f} \chi^g,$$

where  $\mathcal{N}$  is a zero representations of  $\mathcal{A}(\mathbf{GLU})_{n-1}$  of dimension  $\dim(f) - \sum_{g \prec_{\mathbf{GLU}} f} \dim(g)$ .

We need to introduce some additional notations to state an analogue of Theorem 2.26 for  $\mathbf{GLU}$ .

**Definition 7.7.**  $\mathcal{CY}' \subset \mathcal{CY}$  is defined as the set of families  $f$  of Young diagrams such that  $f("x - a") = \emptyset$  for  $a \in \mathbb{F}_q^*$ .

**Definition 7.8.**  $\Omega(\mathbf{GLU})$  is the set of quadruples  $(\alpha, \beta, \gamma, f)$ , where  $\gamma = \{\gamma^j\}$ ,  $j \in \mathbb{F}_q^*$ ,  $\alpha = \{\alpha_i^j\}$ ,  $\beta = \{\beta_i^j\}$ ,  $i = 1, 2, 3, \dots$ ,  $j \in \mathbb{F}_q^*$ ; for every  $j$  the sequences  $\alpha_i^j$  and  $\beta_i^j$  and the number  $\gamma^j$  satisfy (2.2); moreover  $\sum_j \gamma^j = 1$  and  $f \in \mathcal{CY}'$ .

**Theorem 7.9** (Classification theorem for finite traces of  $\mathcal{A}(\mathbf{GLU})$ ). *The extreme rays of the set of traces of  $\mathcal{A}(\mathbf{GLU})$  are parameterized by elements of  $\Omega(\mathbf{GLU})$ . For  $\omega = (\alpha, \beta, \gamma, f) \in \Omega(\mathbf{GLU})$  the corresponding ray is  $\mathbb{R}_+ \chi^\omega(\cdot)$  and for  $g \in \mathbb{GL}(n, q)$  we have  $\chi^\omega(I_g^{\mathbf{GLU}}) = 0$  if  $n < |f|$ , otherwise,*

$$(7.1) \quad \chi^\omega(I_g^{\mathbf{GLU}}) = \sum_{\lambda^j \in \mathbb{Y}: \sum |\lambda^j| = n - |f|} \chi^{f + \sum_j E_j(\lambda^j)}(I_g^{\mathbf{GLU}}) \prod_{j \in \mathbb{F}_q^*} sp_{\alpha^j, \beta^j, \gamma^j}[s_{\lambda^j}],$$

Where  $E_j(\lambda) \in \mathcal{CY}$  is a family taking value  $\lambda$  in “ $x - j$ ” ( $j \in \mathbb{F}_q^*$ ) and taking  $\emptyset$  in all other points.

*Proof.* The argument starts similarly to that of Theorem 2.26. For a family  $f \in \mathcal{CY}'$  let  $\mathcal{CY}^{[f]} \subset \mathcal{CY}$  denote the set of families  $h \in \mathcal{CY}$  such that  $h(u) = f(u)$  for all  $u \in \mathcal{C} \setminus \bigcup_{j \in \mathbb{F}_q^*} \{“x - j”\}$ .

Moreover, for a family  $f \in \mathcal{CY}'$  let  $\Theta^f$  denote the convex cone of traces  $\chi$  of  $\mathcal{A}(\mathbf{GLU})$  such that for  $n < |f|$  the restriction  $\chi|_{\mathcal{A}(\mathbf{GLU})_{n-1}}$  vanishes and for  $n \geq |f|$  in the decomposition

$$\chi|_{\mathcal{A}(\mathbf{GLU})_n} = \sum_{h \in \mathcal{CY}_n} c(h) \chi^h(\cdot).$$

$c(h) = 0$  unless  $h \in \mathcal{CY}^{[f]}$ . Let  $\Theta^\emptyset$  denote the set  $\Theta^f$  for  $f$  being the empty family. Similarly to the proof of Theorem 2.26 for any  $f \in \mathcal{CY}'$  the convex cone  $\Theta^f$  is affine isomorphic to  $\Theta^\emptyset$  and the statement of Theorem 7.9 is reduced to the identification of all extreme rays of  $\Theta^\emptyset$ . The rest of the proof is this identification.

Take  $q - 1$  countable sets of variables  $(x_i^j)_{i=1,2,\dots}$ ,  $j \in \mathbb{F}_q^*$  and let  $\Lambda^{\otimes(q-1)}$  denote the algebra of (polynomial) functions symmetric in variables  $x_i^j$ ,  $i = 1, 2, \dots$  for every fixed  $j \in \mathbb{F}_q^*$ . Let  $\Lambda^j$  denote the subalgebra of symmetric functions in  $x_1^j, x_2^j, \dots$ . For any  $j$  and any symmetric function  $r \in \Lambda$  let  $r(x^j)$  denote the corresponding symmetric function in variables  $x_1^j, x_2^j, \dots$ . In particular,  $p_k(x^j) \in \Lambda^j$  are the Newton power sum in variables  $x^j$

$$p_k(x^j) = \sum_{i=1}^{\infty} (x_i^j)^k.$$

Clearly, the functions

$$\prod_{j \in \mathbb{F}_q^*} s_{\lambda^j}(x^j), \quad \lambda^j \in \mathbb{Y}$$

form a linear basis in  $\Lambda^{\otimes(q-1)}$ .

Let  $\Delta$  denote the cone of linear functionals  $w$  on  $\Lambda^{\otimes(q-1)}$  satisfying:

$$w : \Lambda^{\otimes(q-1)} \rightarrow \mathbb{C}$$

- (1)  $w \left[ \prod_{j \in \mathbb{F}_q^*} s_{\lambda^j}(x^j) \right] \geq 0$  for any  $q - 1$  Young diagrams  $\{\lambda^j\}$ .
- (2)  $w \left[ u \left( \sum_{j \in \mathbb{F}_q^*} p_1(x^j) \right) \right] = w[u]$ , for any  $u \in \Lambda^{\otimes(q-1)}$ .

We claim that  $\Delta$  and  $\Theta^\emptyset$  are affine isomorphic. Under this identification a trace  $\chi \in \Theta^\emptyset$  corresponds to a functional  $w_\chi$  such that

$$w_\chi \left[ \prod_{j \in \mathbb{F}_q^*} s_{h(\text{"}x-j\text{"})}(x^j) \right] = c(h)$$

for any  $h \in \mathcal{CY}^{[\emptyset]}$  and for  $h \in \mathcal{CY}^{[\emptyset]} \cap \mathcal{CY}_n$  the number  $c(h)$  is defined as the coefficient in the decomposition

$$\chi|_{\mathcal{A}(\mathbf{GLU})_n} = \sum_{h \in \mathcal{CY}_n} c(h) \chi^h(\cdot).$$

Indeed, condition 1 translates into the non-negativity of the coefficients  $c(h)$  and condition 2 translates into the statement of Proposition 7.6.

Now we can again use the *Ring Theorem* (see [KV80], [VK90], [K03] and also [GO1, Section 8.7]) for studying the structure of the set  $\Delta$ . This theorem yields that  $h \in \Delta$  is an element of an extreme ray (in other words,  $h$  is indecomposable) if and only if  $h = r\hat{h}$ , where  $r \in \mathbb{R}_+$  and  $\hat{h} \in \Delta$  is a multiplicative functional, i.e.  $\hat{h}(uv) = \hat{h}(u)\hat{h}(v)$ .

Now let  $h \in \Delta$  be a multiplicative functional and let  $h^j, j \in \mathbb{F}_q^*$  be its restrictions on  $\Lambda^j$ . Clearly,

$$(7.2) \quad h \left[ \prod_{j \in \mathbb{F}_q^*} s_{\lambda^j}(x^j) \right] = \prod_{j \in \mathbb{F}_q^*} h^j[s_{\lambda^j}(x^j)]$$

Define a new linear functional  $\hat{h}^j$  on  $\Lambda^j$  through

$$\hat{h}^j[u] := \frac{h^j[u]}{(h^j[p_1(x^j)])^{\deg(u)}},$$

where  $\deg(u)$  is the degree of polynomial  $u$ . The functional  $\hat{h}^j$  on  $\Lambda^j$  is multiplicative and satisfies

- (1)  $\hat{h}^j[s_\lambda(x^j)] \geq 0$  for every  $\lambda \in \mathbb{Y}$ ,
- (2)  $\hat{h}^j[up_1(x_j)] = h^j[u]$ ,
- (3)  $h^j[p_1(x_j)] = 1$ .

Such functional are classified by Thoma's theorem, they correspond to extreme points of the set of normalized characters of  $S(\infty)$ , see [VK90], [K03]. They are parameterized by sequences  $\alpha, \beta$ , satisfying (2.2) with  $\gamma = 1$ . We have:

$$\hat{h}^j[s_\lambda(x^j)] = \mathcal{S}p_{\alpha, \beta, 1}[s_\lambda].$$

Now set  $\gamma_j = h^j[p_1(x^j)]$ . Then we have

$$h^j[s_\lambda(x^j)] = \mathcal{S}p_{\alpha^j, \beta^j, \gamma^j}[s_\lambda]$$

for certain  $\alpha^j, \beta^j, \gamma^j$ . Substituting into (7.2) we arrive at the desired statement.  $\square$

Theorem 7.9 yields that for **GLU** and  $\omega = (\alpha, \beta, f) \in \Omega(\mathbf{GLU})$  the trace  $\chi^\omega$  is unipotent if  $f \equiv \emptyset$  and  $\alpha_i^j = \beta_i^j = 0$  for  $j \neq 1$ . Note that if we identify  $\mathcal{A}(\mathbf{GLB})_n \simeq \mathbb{C}(\mathbf{GL}(n, q)) \simeq \mathcal{A}(\mathbf{GLU})_n$ , then unipotent traces of  $\mathcal{A}(\mathbf{GLB})$  and  $\mathcal{A}(\mathbf{GLU})$  are *the same* functions.

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